**Two characterizations of weak amenability of commutative Banach algebras**

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# ***Abstract***

## In this paper, we shall describe two nice characterizations of weak amenability of commutative Banach algebras. We shall prove the equivalence between Groenbaek's characterization, and Johnson's characterization for these algebras.

# ***Keywords***

## Banach algebras, commutative Banach algebras, weak amenability.

**1. Introduction**

We follow [1] to recall some definitions and some preliminaries. Let $A$ be a Banach algebra, and let $X$ be a Banach $A$-bimodule. A linear map $D:A\rightarrow X$ is a *derivation* if it satisfies the equation:

 $D(ab)=a⋅Db+Da⋅b (a,b\in A) .$

In this paper we shall only consider bounded derivations. Given $x\in X$ and define the map $δ\_{x}:A\rightarrow X$ by the equation:

 $δ\_{x}(a)=a⋅x-x⋅a (a\in A).$

These derivations are *inner* derivations.

Let $X^{\*}$ be the *dual space* of $X$. Then $X^{\*}$ is a Banach $A$-bimodule with respect to the operations given by

 $〈x,a⋅λ〉=〈x⋅a,λ〉 and 〈x,λ⋅a〉=〈a⋅x,λ〉 (a\in A,x\in X,λ\in X^{\*}) .$

A Banach algebra $A$ is *amenable* if every bounded derivation $D$ from $A$ into a dual Banach $A$-bimodule $X^{\*}$ is inner, for each Banach $A$-bimodule $X$.

A Banach algebra $A$ is a Banach $A$-bimodule over itself. Then a Banach algebra $A$ is *weakly amenable* if every bounded derivation $D:A\rightarrow A^{\*}$ is inner.

Let $A$ be a Banach algebra, $A^{\#}$ denotes the Banach algebra formed by adjoining an identity to $A$.

Let $A$ be a Banach algebra, and suppose that $E$ and $F$ are Banach left and right $A$-modules, respectively. Then $E\hat{⊗}F$ is a Banach $A$-bimodule for unique products that satisfy

 $a⋅(x⊗y)=a⋅x⊗y , (x⊗y)⋅a=x⊗y⋅a (a\in A, x\in E, y\in F) .$

In particular, $A\hat{⊗}A^{ \#}$ is a Banach $A$-bimodule.

There is a bounded linear map $π:A\hat{⊗}A^{ \#}\rightarrow A$ such that

 $π(a⊗b)=ab (a\in A,b\in A^{\#}) ;$

 this map is called *the induced product map*. We see that

 $kerπ=\left\{\sum\_{i=1}^{\infty }‍a\_{i}⊗b\_{i}\in A\hat{⊗}A^{ \#}:\sum\_{i=1}^{\infty }‍a\_{i}b\_{i}=0\right\}$ (1.1)

Sometimes we shall write $K\_{A}$ for $kerπ$; it is a closed Banach sub-bimodule in $A\hat{⊗}A^{ \#}$.

**Lemma 1.1** *Let* $A$ *be a unital Banach algebra. Then*

 $ker π=\overline{lin}\left\{a⊗b-c⊗d:ab=cd\right\} ,$ (1.2)

 where, in the right-hand side, $a,c\in A$ and $b,d\in A^{ \#}$ .$∎$

**2. Basic algebraic characterization of weak amenability**

In his paper [3] Johnson established a characterization for a Banach algebra to be weakly amenable, as follows.

Let $A$ be a Banach algebra. We define the bounded linear maps

 $D\_{1}:A\hat{⊗}A\rightarrow A andD\_{2}:A\hat{⊗}A\hat{⊗}A\rightarrow A\hat{⊗}A$

to be such that

 $D\_{1}(a⊗b)=ab-ba andD\_{2}(a⊗b⊗c)=a⊗bc-ab⊗c+b⊗ca .$

for all $a,b,c\in A .$ These maps exist by basic properties of the projective tensor product, and clearly $\left‖D\_{1}\right‖\leq 2$ .

It can be seen that the composition map $D\_{1}∘D\_{2}$ is zero, and so $\overline{im}D\_{2}⊂kerD\_{1}$.

Suppose that $D$ is a bounded linear map from $A$ to $A^{\*}$, and define the corresponding linear functional $F\_{D}$ in $(A\hat{⊗}A)^{\*}$ to be such that

 $F\_{D}(a⊗b)=\left⟨b,D(a)\right⟩ (a,b\in A) .$

Then, for all elements $a,b$ and $c$ in $A$, we have

 $\left(F\_{D}∘D\_{2}\right)(a⊗b⊗c)=F\_{D}(a⊗bc)-F\_{D}(ab⊗c)+F\_{D}(b⊗ca)$

 $=\left⟨bc,D(a)\right⟩-\left⟨c,D(ab)\right⟩+\left⟨ca,D(b)\right⟩$

 $=\left⟨c,D(a)⋅b\right⟩-\left⟨c,D(ab)\right⟩+\left⟨c,a⋅D(b)\right⟩$

 $=\left⟨c,D(a)⋅b-D(ab)+a⋅D(b)\right⟩ .$

Thus we have that $F\_{D}∘D\_{2}=0$ if and only if $D$ is a derivation.

Moreover, let $λ\in A^{\*}$. Then the above derivation $D$ is the inner derivation determined by $λ$ if and only if $F\_{D}=λ∘D\_{1}$. To see this, first set $D=δ\_{λ} ,$ and take $a,b\in A$ . Then

 $F\_{D}(a⊗b)=\left⟨b,δ\_{λ}(a)\right⟩=\left⟨b,a⋅λ-λ⋅a\right⟩=\left⟨ba-ab,λ\right⟩ ,$

 and $\left(λ∘D\_{1}\right)(a⊗b)=\left⟨ba-ab,λ\right⟩$ . Thus $F\_{D}=λ∘D\_{1}$ .

Conversely, suppose that $F\_{D}=λ∘D\_{1} .$ Then

 $\left⟨b,D(a)\right⟩=\left⟨ab-ba,λ\right⟩=\left⟨b,λ⋅a-a⋅λ\right⟩ (a,b\in A)$

 and so $D(a)=λ⋅a-a⋅λ=δ\_{λ}(a) (a\in A) ,$ and hence $D=δ\_{λ}$ is inner.

We write $Z=imD\_{1}$ , so that $Z$ is a subspace of $A$; we also write $\left‖ ⋅ \right‖\_{π}$ for the quotient of the projective norm from $A\hat{⊗}A$ on $Z$. Thus we see that

 $\left‖z\right‖\leq \left‖D\_{1}\right‖\left‖z\right‖\_{π}\leq 2\left‖z\right‖\_{π} (z\in Z) .$ (2.1)

The following theorem is stated by Johnson in [3]. However, no proof is given in [3], and we do not find the result to be immediate, and so we offer a proof.

**Theorem 2.1**  *(Johnson) Let* $\left(A,\left‖ ⋅ \right‖\right)$ *be a Banach algebra. Then* $A$ *is weakly amenable if and only if* $im D\_{1}$ *is closed and* $\overline{im}D\_{2}=ker D\_{1}$ *in* $A\hat{⊗}A$ *.*

**Proof** Suppose that $A$ is weakly amenable.We shall first prove that $Z=imD\_{1}$ is closed in $A$.

Let $λ$ be a continuous linear functional on $(Z,\left‖ ⋅ \right‖\_{π}) ,$ and define

 $\left⟨b, D\_{λ}(a)\right⟩=\left⟨ab-ba,λ\right⟩ (a,b\in A) .$ (2.2)

Clearly $D\_{λ}(a)$ is a linear functional on $A$, and

 $\left|\left⟨b, D\_{λ}(a)\right⟩\right|\leq \left‖λ\right‖\left‖ab-ba\right‖\_{π}\leq 2\left‖λ\right‖\left‖a\right‖\left‖b\right‖ (a,b\in A) ,$ (2.3)

 so that $D\_{λ}(a)\in A^{\*}$ and $\left‖D\_{λ}(a)\right‖\leq 2\left‖λ\right‖\left‖a\right‖$ for $a\in A)$ , and hence the map $D\_{λ}:A\rightarrow A^{\*}$ is a continuous linear map. It is easily checked that $D\_{λ}$ is a derivation.

Since $A$ is weakly amenable, there is a continuous linear functional $μ\in A^{\*}$ such that

 $D\_{λ}(a)=a ⋅ μ-μ ⋅ a (a\in A) .$

Clearly, we have

 $\left⟨ab-ba,λ\right⟩=\left⟨ba-ab, μ\right⟩ (a,b\in A)$

and $\left|\left⟨z,μ\right⟩\right|\leq 2\left‖μ\right‖\left‖z\right‖\_{π}(z\in Z)$ by (2.1) , so $μ|\_{Z}$ is continuous on $Z,\left‖ ⋅ \right‖\_{π}$ .

Since $λ$ and $μ$ are continuous linear functionals on $(Z,\left‖ ⋅ \right‖\_{π})$ which agree at elements $ab-ba$ for all $a,b,\in A$ , we have $λ=μ$ on $Z$ . Hence the map $Θ:μ↦μ|\_{Z}$ , $A^{\*}\rightarrow \left(Z,\left‖ ⋅ \right‖\_{π}\right)^{\*}$ is a continuous surjection, and so by the open mapping theorem there is a constant $C$ such that, for each $λ\in \left(Z,\left‖ ⋅ \right‖\_{π}\right)^{\*}$ , there exists $μ\in A^{\*}$ with $μ|\_{Z}=λ$ and $\left‖μ\right‖\leq C\left‖λ\right‖$ . Thus $\left‖z\right‖\_{π}\leq C\left‖z\right‖ (z\in Z)$ , and so $Z$ is closed in $A$ .

Now, we shall prove that $\overline{im}D\_{2}=kerD\_{1}$ in $A\hat{⊗}A$ .

Since $D\_{1}∘D\_{2}=0 ,$ we see that $\overline{im}D\_{2}⊂kerD\_{1}$ , so that it is enough to prove that $kerD\_{1}⊂\overline{im}D\_{2} .$

Take $μ\in (A\hat{⊗}A)^{\*}$ with $μ|\overline{im}D\_{2}=0 .$ Define a map $D:A\rightarrow A^{\*}$ by

 $\left⟨b,D(a)\right⟩=\left⟨a⊗b,μ\right⟩ (a,b\in A) .$

Then $D$ is a continuous linear map .

We *claim* that $D$ is a derivation. Indeed, for $a,b,c\in A$ , we have

$$\left⟨c,D(ab)\right⟩-\left⟨c,a⋅D(b)\right⟩-\left⟨c,D(a)⋅b\right⟩=\left⟨c,D(ab)\right⟩-\left⟨ca,D(b)\right⟩-\left⟨bc,D(a)\right⟩$$

 $=\left⟨ab⊗c-b⊗ca-a⊗bc,μ\right⟩=0$

 because $μ|\overline{im}D\_{2}=0$ , and so the claim holds.

Since $A$ is weakly amenable, $D$ is an inner derivation, say $D=δ\_{λ}$ for some $λ\in A^{\*}$ . But now

 $\left⟨a⊗b,μ\right⟩=(λ∘D\_{1})(a⊗b) (a,b\in A) ,$

 and so $μ=λ∘D\_{1}$ . In particular, $μ|kerD\_{1}=0$ . By the Hahn-Banach theorem, $\overline{im}D\_{2}=kerD\_{1}$ .

Conversely, suppose that $Z$ is closed in $A$ and $\overline{im}D\_{2}=kerD\_{1} .$

To show that $A$ is weakly amenable, take a bounded derivation $D:A\rightarrow A^{\*} .$ As above, we have $F\_{D}∘D\_{2}=0$ . Thus $F\_{D}|\overline{im}D\_{2}=0 .$ But $\overline{im}D\_{2}=kerD\_{1} ,$ and so $F\_{D}|kerD\_{1}=0 .$ This show that $F\_{D}$ defines a continuous linear functional on $(A\hat{⊗}A)/kerD\_{1}=imD\_{1}=\left(Z,\left‖ ⋅ \right‖\_{π}\right) .$ Since $Z$ is closed in $A$ , the norms $\left‖ ⋅ \right‖$ and $\left‖ ⋅ \right‖\_{π}$ are equivalent on $Z$ . Hence $F\_{D}$ is also a continuous linear functional on $\left(Z,\left‖ ⋅ \right‖\right)$. By the Hahn–Banach theorem, $F\_{D}$ extends to $λ\in A^{\*} .$ We have $F\_{D}=λ∘D\_{1}$ on $A\hat{⊗}A ,$ and so $D$ is the inner derivation determined by $λ .$ Therefore $A$ is weakly amenable.Thus the theorem is proved. $∎$

Note that in the commutative case, we see that $D\_{1}=0$, and so $A$ is weakly amenable if and only if $\overline{im}D\_{2}=A\hat{⊗}A$ .

In the following theorem, we present the Groenbaek’s characterization of weak amenability of commutative Banach algebras. Let $A$ be a commutative Banach algebra. We recall the subspace $K\_{A}$ of $A\hat{⊗}A^{\#}$ was defined in (1.1).

**Theorem 2.2**  *(Groenbaek) Let* $A$ *be a commutative Banach algebra. Then* $A$ *is weakly amenable if and only if* $\overline{K\_{A}^{2}}=K\_{A}$*.* $∎$

**3. The equivalence of two characterizations**

In this section, we shall deal with commutative Banach algebras. We shall prove a nice equivalence between Groenbaek’s and Johnson’s characterizations; we continue to use same notions and symbols that we mentioned in the previous section. In fact, we restrict to unital algebras.

**Theorem 3.1** *Let* $A$ *be a unital, commutative Banach algebra. Then the following are equivalent:*

(i)$A$ is weakly amenable;

(ii)$\overline{K\_{A}^{2}}=K\_{A}$ ;

(iii)$\overline{im}D\_{2}=A\hat{⊗}A$ .

**Proof** By Theorem 2.2, (i) and (ii) are equivalent. By Theorem 2.1 (in the commutative case), (i) and (iii) are equivalent. We shall give a direct proof that (ii) and (iii) are equivalent, thus giving a proof that is independent of Theorem 2.2 .

We write $K$ for $K\_{A}$ , and $e\_{A}$ for the identity of $A$ .

First we suppose that (ii) holds, so that $\overline{K^{2}}=K$. We shall show that $\overline{im}D\_{2}=A\hat{⊗}A$, and hence obtain (iii).

Set $Λ=\overline{im}D\_{2}$, so that

 $Λ=\overline{lin}\left\{a⊗bc-ab⊗c+b⊗ca:a,b,c\in A\right\} .$ (3.1)

By taking $b=e\_{A}$ , we see that $e\_{A}⊗ca\in Λ$ for each $a,b,c\in A$ .

Take $φ\in \left(A\hat{⊗}A\right)^{\*}$ such that $φ|\_{Λ}=0$ .

Let $a,b,c\in A$ . Then

 $φ(e\_{A}⊗ca)=0 .$ (3.2)

Similarly, putting $c=e\_{A}$ in (3.1), we have $a⊗b-ab⊗e\_{A}+b⊗a$ in $Λ$ , and so

 $φ(a⊗b)+φ(b⊗a)=φ(ab⊗e\_{A}) .$ (3.3)

Now we have

 $(e\_{A}⊗a-a⊗e\_{A})(e\_{A}⊗b-b⊗e\_{A})=e\_{A}⊗ab+ab⊗e\_{A}-a⊗b-b⊗a ,$

so that, from (3.2) and (3.3), we have

 $φ\left((e\_{A}⊗a-a⊗e\_{A})(e\_{A}⊗b-b⊗e\_{A})\right)=φ(e\_{A}⊗ab)+φ(ab⊗e\_{A})$

 $-φ(a⊗b)-φ(b⊗a)=0 .$

We now *claim* that $φ(\overline{K^{2}})=0$ . By Lemma 1.1, it is sufficient to show that

 $φ\left((a⊗b-a^{'}⊗b^{'})(u⊗v-u^{'}⊗v^{'})\right)=0$

whenever $a,a^{'},u,u^{'},b,b^{'},v,v^{'}\in A$ with $ab=a^{'}b^{'}=m\_{1}$ and $uv=u^{'}v^{'}=m\_{2}$ . Then $(a⊗b-a^{'}⊗b^{'})(u⊗v-u^{'}⊗v^{'})$ is an element in $K^{2}$ , and, by using the following equalities:

 $φ(au⊗bv)=φ(a⊗ubv)+φ(u⊗bva)=φ(a⊗m\_{2}b)+φ(u⊗m\_{1}v) ;$

 $φ(a^{'}u⊗b^{'}v)=φ(a^{'}⊗ub^{'}v)+φ(u⊗b^{'}va^{'})=φ(a^{'}⊗m\_{2}b^{'})+φ(u⊗m\_{1}v) ;$

 $φ(au^{'}⊗bv^{'})=φ(a⊗u^{'}bv^{'})+φ(u^{'}⊗bv^{'}a)=φ(a⊗m\_{2}b)+φ(u^{'}⊗m\_{1}v^{'}) ;$

$$φ(a^{'}u^{'}⊗b^{'}v^{'})=φ(a^{'}⊗u^{'}b^{'}v^{'})+φ(u^{'}⊗b^{'}v^{'}a^{'})=φ(a^{'}⊗m\_{2}b^{'})+φ(u^{'}⊗m\_{1}v^{'}) ;$$

 we see that

 $φ\left((a⊗b-a^{'}⊗b^{'})(u⊗v-u^{'}⊗v^{'})\right)=φ(au⊗bv)-φ(a^{'}u⊗b^{'}v)$

 $-φ(au^{'}⊗bv^{'})+φ(a^{'}u^{'}⊗b^{'}v^{'})=0 .$

Thus $φ(\overline{K^{2}})=0$ as required. It follows from (ii), that $φ|\_{K}=0$ because $\overline{K^{2}}=K$ .

Now take $z=\sum\_{i=1}^{\infty }‍a\_{i}⊗b\_{i}\in A\hat{⊗}A$ , with $\sum\_{i=1}^{\infty }‍\left‖a\_{i}\right‖\left‖b\_{i}\right‖<\infty $ . Then

 $z=\sum\_{i=1}^{\infty }‍\left(a\_{i}⊗b\_{i}-e\_{A}⊗a\_{i}b\_{i}\right)+\sum\_{i=1}^{\infty }‍e\_{A}⊗a\_{i}b\_{i}\in K+Λ ,$

because, for each $i\in N$ , we have $a\_{i}⊗b\_{i}-e\_{A}⊗a\_{i}b\_{i}\in K$ , and $e\_{A}⊗a\_{i}b\_{i}\in Λ ,$ as we remarked, and because both $Λ$ and $K$ are closed. So $φ(z)=0$ . Hence $φ=0$ . By the Hahn–Banach theorem, $Λ=A\hat{⊗}A$ , as required.

We now suppose that (iii) holds, so that $\overline{im}D\_{2}=A\hat{⊗}A$. We need to prove that $\overline{K^{2}}=K$.

Let $a,b,c\in A .$ Then

$$a⊗bc-ab⊗c+b⊗ca=e\_{A}⊗abc-(e\_{A}⊗a-a⊗e\_{A})(e\_{A}⊗b-b⊗e\_{A})e\_{A}⊗c\in e\_{A}⊗abc+K^{2} .$$

Now take $x\in K ,$ so that $π(x)=0 .$ Since $\overline{im}D\_{2}=A\hat{⊗}A$ , we can write $x=lim\_{n\rightarrow \infty }r\_{n} ,$ say, where $r\_{n}\in imD\_{2} (n\in N)$ . We have $π(r\_{n})\rightarrow 0$ as $n\rightarrow \infty .$

By the definition of $D\_{2}$ , we may write

 $r\_{n}=\sum\_{i=1}^{\infty }‍\left(a\_{n,i}⊗b\_{n,i}c\_{n,i}+b\_{n,i}⊗a\_{n,i}c\_{n,i}-a\_{n,i}b\_{n,i}⊗c\_{n,i}\right) ,$

where $\sum\_{i=1}^{\infty }‍\left‖a\_{n,i}\right‖\left‖b\_{n,i}\right‖\left‖c\_{n,i}\right‖<\infty $ . Thus we have

$$r\_{n}=\sum\_{i=1}^{\infty }‍\left(e\_{A}⊗a\_{n,i}b\_{n,i}c\_{n,i}+(e\_{A}⊗a\_{n,i}-a\_{n,i}⊗e\_{A})(e\_{A}⊗b\_{n,i}-b\_{n,i}⊗e\_{A})e\_{A}⊗c\_{n,i}\right)$$

$$=e\_{A}⊗π(r\_{n})+\sum\_{i=1}^{\infty }‍\left((e\_{A}⊗a\_{n,i}-a\_{n,i}⊗e\_{A})(e\_{A}⊗b\_{n,i}c\_{n,i}-b\_{n,i}⊗c\_{n,i}\right)$$

 because $π(r\_{n})=\sum\_{i=1}^{\infty }‍a\_{n,i}b\_{n,i}c\_{n,i}$ and because $A$ is commutative.

Thus we have $r\_{n}\in e\_{A}⊗π(r\_{n})+\overline{K^{2}} .$

But $π(r\_{n})$ converges to zero as $n\rightarrow \infty $ , so

 $x=\lim\_{n\to \infty }r\_{n}=\lim\_{n\to \infty }(r\_{n}-e\_{A}⊗π(r\_{n}))\in \overline{K^{2}} .$

Therefore $K⊂\overline{K^{2}} ,$ and so (ii) follows.

Thus the theorem is proved. $∎$

In fact, by a small extra argument, the result holds even when $A$ does not have an identity.

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