**Two characterizations of weak amenability of commutative Banach algebras**

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# ***Abstract***

## In this paper, we shall describe two nice characterizations of weak amenability of commutative Banach algebras. We shall prove the equivalence between Groenbaek's characterization, and Johnson's characterization for these algebras.

# ***Keywords***

## Banach algebras, commutative Banach algebras, weak amenability.

**1. Introduction**

We follow [1] to recall some definitions and some preliminaries. Let be a Banach algebra, and let be a Banach -bimodule. A linear map is a *derivation* if it satisfies the equation:

In this paper we shall only consider bounded derivations. Given and define the map by the equation:

These derivations are *inner* derivations.

Let be the *dual space* of . Then is a Banach -bimodule with respect to the operations given by

A Banach algebra is *amenable* if every bounded derivation from into a dual Banach -bimodule is inner, for each Banach -bimodule .

A Banach algebra is a Banach -bimodule over itself. Then a Banach algebra is *weakly amenable* if every bounded derivation is inner.

Let be a Banach algebra, denotes the Banach algebra formed by adjoining an identity to .

Let be a Banach algebra, and suppose that and are Banach left and right -modules, respectively. Then is a Banach -bimodule for unique products that satisfy

In particular, is a Banach -bimodule.

There is a bounded linear map such that

this map is called *the induced product map*. We see that

(1.1)

Sometimes we shall write for ; it is a closed Banach sub-bimodule in .

**Lemma 1.1** *Let be a unital Banach algebra. Then*

(1.2)

where, in the right-hand side, and .

**2. Basic algebraic characterization of weak amenability**

In his paper [3] Johnson established a characterization for a Banach algebra to be weakly amenable, as follows.

Let be a Banach algebra. We define the bounded linear maps

to be such that

for all These maps exist by basic properties of the projective tensor product, and clearly .

It can be seen that the composition map is zero, and so .

Suppose that is a bounded linear map from to , and define the corresponding linear functional in to be such that

Then, for all elements and in , we have

Thus we have that if and only if is a derivation.

Moreover, let . Then the above derivation is the inner derivation determined by if and only if . To see this, first set and take . Then

and . Thus .

Conversely, suppose that Then

and so and hence is inner.

We write , so that is a subspace of ; we also write for the quotient of the projective norm from on . Thus we see that

(2.1)

The following theorem is stated by Johnson in [3]. However, no proof is given in [3], and we do not find the result to be immediate, and so we offer a proof.

**Theorem 2.1**  *(Johnson) Let be a Banach algebra. Then is weakly amenable if and only if is closed and in .*

**Proof** Suppose that is weakly amenable.We shall first prove that is closed in .

Let be a continuous linear functional on and define

(2.2)

Clearly is a linear functional on , and

(2.3)

so that and for , and hence the map is a continuous linear map. It is easily checked that is a derivation.

Since is weakly amenable, there is a continuous linear functional such that

Clearly, we have

and by (2.1) , so is continuous on .

Since and are continuous linear functionals on which agree at elements for all , we have on . Hence the map , is a continuous surjection, and so by the open mapping theorem there is a constant such that, for each , there exists with and . Thus , and so is closed in .

Now, we shall prove that in .

Since we see that , so that it is enough to prove that

Take with Define a map by

Then is a continuous linear map .

We *claim* that is a derivation. Indeed, for , we have

because , and so the claim holds.

Since is weakly amenable, is an inner derivation, say for some . But now

and so . In particular, . By the Hahn-Banach theorem, .

Conversely, suppose that is closed in and

To show that is weakly amenable, take a bounded derivation As above, we have . Thus But and so This show that defines a continuous linear functional on Since is closed in , the norms and are equivalent on . Hence is also a continuous linear functional on . By the Hahn–Banach theorem, extends to We have on and so is the inner derivation determined by Therefore is weakly amenable.Thus the theorem is proved.

Note that in the commutative case, we see that , and so is weakly amenable if and only if .

In the following theorem, we present the Groenbaek’s characterization of weak amenability of commutative Banach algebras. Let be a commutative Banach algebra. We recall the subspace of was defined in (1.1).

**Theorem 2.2**  *(Groenbaek) Let be a commutative Banach algebra. Then is weakly amenable if and only if .*

**3. The equivalence of two characterizations**

In this section, we shall deal with commutative Banach algebras. We shall prove a nice equivalence between Groenbaek’s and Johnson’s characterizations; we continue to use same notions and symbols that we mentioned in the previous section. In fact, we restrict to unital algebras.

**Theorem 3.1** *Let be a unital, commutative Banach algebra. Then the following are equivalent:*

(i) is weakly amenable;

(ii) ;

(iii) .

**Proof** By Theorem 2.2, (i) and (ii) are equivalent. By Theorem 2.1 (in the commutative case), (i) and (iii) are equivalent. We shall give a direct proof that (ii) and (iii) are equivalent, thus giving a proof that is independent of Theorem 2.2 .

We write for , and for the identity of .

First we suppose that (ii) holds, so that . We shall show that , and hence obtain (iii).

Set , so that

(3.1)

By taking , we see that for each .

Take such that .

Let . Then

(3.2)

Similarly, putting in (3.1), we have in , and so

(3.3)

Now we have

so that, from (3.2) and (3.3), we have

We now *claim* that . By Lemma 1.1, it is sufficient to show that

whenever with and . Then is an element in , and, by using the following equalities:

we see that

Thus as required. It follows from (ii), that because .

Now take , with . Then

because, for each , we have , and as we remarked, and because both and are closed. So . Hence . By the Hahn–Banach theorem, , as required.

We now suppose that (iii) holds, so that . We need to prove that .

Let Then

Now take so that Since , we can write say, where . We have as

By the definition of , we may write

where . Thus we have

because and because is commutative.

Thus we have

But converges to zero as , so

Therefore and so (ii) follows.

Thus the theorem is proved.

In fact, by a small extra argument, the result holds even when does not have an identity.

**References**

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