Modeling of Annual Maximum Temperature Using Weibull Distribution With Application:
A Case Study in Sebha City

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Abstract

Weibull distribution appears very frequently in practical when we observed data that represent extreme values. In this paper, the first objective is to identify the domain of attraction of a given distribution to belong in the domain attraction of maximum Weibull distribution by using Castillo and de Haan, necessary and sufficient condition. The second objective is used diagnostics plot by using three different plots, probability paper (P-P) plot, Quantile Quantile (Q-Q) plot and Return level (R-L) plot to detect an appropriate distribution of extreme. The third objective of the present study is to model the behavior of annual maximum temperature data by using maximum Weibull distribution. Maximum Likelihood Estimation (MLE) is used to estimate the parameters of distribution. The results show that the maximum temperature is significant to be fitted by **max-Weibull** model and it is the better choice basis on LR-test and graphical diagnostics. For the parameter estimation, LR-test and diagnostic plots calculations, we use the R programming language with packages of fExtreme and ismev to perform these calculations.

I. INTRODUCTION

Extreme value theory (EVT) is unique as a statistical discipline in that it develops techniques and models for describing the unusual rather than the usual. Extreme value analysis (EVA), differs from other typical statistical techniques and it is based on the analysis of the maximum (or minimum) value in a selected time period. EVA has been used widely in many areas of application ranging from insurance and finance to meteorology and hydrology. Early references include the work by[1]., who identified one possible limit distribution for maximum. Soon after, [1] showed that extreme limit laws can be only one of three types by [2] provided simple and useful sufficient conditions for the weak convergence of maximum to each of the three types of limit distributions. In [3], established a rigorous foundation of the EVT when he provided necessary and sufficient conditions for the weak convergence of the sample extremes. His work was refined by [4]. More recent works that offer an alternative viewpoint from the present text are provided by [5]. The latter also includes software for carrying out a range of extreme value analyses by [7], like this book, aims to give an elementary survey of EVT and practice. In the last two decades, there has been an increasing interest in building statistical models for estimating the probability of rare and extreme events, these models, involving EVT, are of a great interest in environmental sciences, engineering, finance, insurance and many other disciplines see,[8,9]. In general, EVT usually requires estimation of the probability of events that are more extreme than any that have already been observed. EVT has been widely used and studied by many researchers. In [10], presents some statistical application of the extreme value theory. There are many areas where EVT plays an important role; see, for example, [11,12,13,14]. This paper is organized as follows: In section 2, describes a brief review of some theoretical results concerning on max-Weibull distribution. Maximum domain of attraction condition and norming constants of max-Weibull distribution are given in section 3. While, in section 4, model estimation is addressed. In section 5, model diagnostic selecting by using graphical methods and hypothesis testing are given. In section 6, the case study of maximum temperature to model the behaviour of extreme by using **max-Weibull** distribution is described. Results and discussion are given in section,7. Finally, concluding remarks and future work are given in section 8.

2 Max-Weibull distribution (W_M)

In EVT, max -Weibull distribution is one of the probability distributions of extreme value and widely used in several areas for modelling of extreme events such as: flood, rainfall, temperature and wind speeds etc. [15,16,17]. max-Weibull distribution is a special case of the generalized extreme value (GEV) distribution when shape parameter $\gamma < 0$. The probability density function (pdf) of max -Weibull ($W_{\max}$) is given by:

$$f(x) = \frac{\gamma}{\sigma}\left(\frac{\mu - x}{\sigma}\right)^{\gamma-1} \exp[-(\frac{\mu - x}{\sigma})^{\gamma}],\ x > \mu \qquad 1$$

And the cumulative distribution function (cdf) is :

$$F(x) = 1 - \exp[-(\frac{x-\mu}{\sigma})^{\gamma}],\ x \geq \mu \qquad 2$$

Where μ, σ, γ are location, scale and shape parameter respectively. The shape parameter $\gamma = 1/\alpha$ is called the extreme value index (EVI).

3 Domain Attraction of max-Weibull Distribution (DAFD)

An interesting problem from the point of view of extremes is about knowing the domain of attraction condition of a given cdf F . To identify the domain of attraction of a given max-Weibull distribution F and the norming constant, we give two theorems that allow solving this problem. Before that, we will present definition and basic properties that provide most of the background necessary to fully understand the mathematical derivations for the distribution F to belong to domain attraction of **max-Weibull** distribution.

3.1 Definition and basic properties

Let F be a distribution function and $Q(y) = \inf\{x : F(x) > y\}$ is the quantile function of F . We define the tail quantile function U by $U(t) = \inf\{x : F(x) \geq 1 - 1/t\}$. Note that, the tail quantile function U and the quantile function Q are linked via the relation $U(t) = Q(1-1/t)$. Let the left and right end point respectively, $v(F) = \inf\{x : F(x) > 0\}$. $w(F) = Sup\{x : F(x) < 1\}$. The relationship between the tail quantile function and the left and right end point written as follows:

$$v(F) = \inf\{x : F(x) > 0\} = U(1) = F^{-1}(0)$$
$$w(F) = Sup\{x : F(x) < 1\} = U(t = \infty) = F^{-1}(1)$$

3.2 Determining the Maximum Domain of Attraction of a CDF (MDA, $F \in D_{\max}(W_M)$)

An interesting problem from the point of view of extremes is about knowing the domain of attraction of a given cdf $F(x)$. To identify the domain of attraction of a given distribution $F(x)$) and the associated sequences a_n , and b_n , we give two theorems that allow solving this problem, see,[10,12,15]. We now present a detailed discussion for cd F to be in the domain attraction of max-Weibull distribution by introducing in the following two theorems of : $F \in D_{\max}(W_M)$

Theorems-1: Castillo and Hadi of sufficient condition for MDA

The distribution function F is in the max-domain attraction of Weibull distribution and we write $F \in D_{\max}(W_M)$ if and only if

$$\lim_{e \to 0} \frac{F^{-1}(1-e) - F^{-1}(1-2e)}{F^{-1}(1-2e) - F^{-1}(1-4e)} = 2^{-\gamma}, \gamma > 0 \qquad 3$$

Where e is the base of the natural logarithm and γ is the shape parameter. In [6], it shows that the last one of necessary and sufficient conditions in term of U have been given in the tail quantile function of a cdf of F .

$$a_n = w(F) = F^{-1}(1)$$

and

$$b_n = w(F) - F^{-1}(1 - n^{-1}) = x^r - U(n) \qquad 4$$

Theorem-2: de Haan and Ferreira of sufficient conditions for $F \in D_{\max}(W_M)$

A cd F is in the Max-domain attraction of Fréchet distribution with $\gamma > 0$ and we write $F \in D_{\max}(F_M)$ if and only if

$$\lim_{t\to\infty}\frac{u(\infty)-u(tx)}{u(\infty)-u(t)} = x^{\gamma} \quad 5$$

A distribution function F is said to be in the domain of max -Weibull distribution and we write $F \in D_{\max}(W_M)$. Finally, the possible choice for norming constants, location a_n and scale b_n, these constants are not unique, depend on the type of domain attraction. We can be chosen the location and scale in the domain of max-Weibull as follows:

3.3 Norming constant of domain of max-Weibull

Under the conditions of $F \in D_{\max}(W_M)$ then**,** the norming constant the location and scale are defined as follows:

$$a_n = w(F) = F^{-1}(1)$$

and $b_n = w(F) - F^{-1}(1-n^{-1}) = x^r - U(n)$ 6

For more details about the norming constants scale and location, see,[7,10].

3.4 Theoretical Examples: In this section, we will give a number of examples to understand the domain attraction **of max-Weibull** distribution, that illustrate in [15], how to find the domain of attraction. We want to introduce two examples, the first one examples belong and the other one example do not belong to max-Weibull distribution. We begin with a simple example that can easily be generalized.

Example-1: Let X be a random variable with uniform distribution $U(0,1)$ has cdf: $F(X) = x \;\; ; 0 \le x \le 1$. For determining the domain attraction of min-Weibull . Since $x^r = F^{-1}(1) = 1$ then we need to determine the sufficient condition of Castillo:

$$F(x) = x \quad ; 0 \le x \le 1 \quad \Rightarrow \quad F^{-1}(x) = x$$

$$\lim_{e\to 0}\frac{F^{-1}(1-e)-F^{-1}(1-2e)}{F^{-1}(1-2e)-F^{-1}(1-4e)} = 2^{-\gamma}, \gamma > 0$$

$$\lim_{e\to 0}\frac{(1-e)-(1-2e)}{(1-2e)-(1-4e)}$$

$$\lim_{e\to 0}\frac{e}{2e} = 2^{-1}, \Rightarrow \gamma = 1$$

In this case we say that F is in the domain attraction of max-Weibull distribution and the notation is $F \in D_{\max}(W_M)$. Also we will try to check whether the **de Haan** sufficient condition (Eq-5) holds. For that purpose, let us note that:

$$\lim_{t\to\infty}\frac{u(\infty)-u(tx)}{u(\infty)-u(t)} = x^{\gamma}$$

$$u(t=\infty) = Q(1-1/t) = 1$$

$$u(t) = Q(1-1/t) = 1-1/t$$

$$u(tx) = Q(1-1/tx) = 1-1/tx$$

$$\lim_{t\to\infty}\frac{u(\infty)-u(tx)}{u(\infty)-u(t)} = x^{\gamma}$$

$$\lim_{t\to\infty}\frac{1-1+1/tx}{1-1+1/t} = \frac{1}{x} = x^{-1}$$

This means that the condition of de Haan hold and we conclude that $F \in D_{\min}(W_m)$ When the domain of attraction condition is satisfied; we can choose the location and scale in domain attraction of min-Weibull as follows:

$$a_n = w(F) = F^{-1}(1) = 1$$

And

$$b_n = w(F) - F^{-1}(1-n^{-1}) = \frac{1}{n}$$

Example-2: **(Cauchy distribution: Minima).** Let X be a random variable with Cauchy distribution has cdf of the Cauchy distribution is

$$F(x) = \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}, \; -\infty < x < \infty$$

and its inverse (quantile) function is:

$$x_p = F^{-1}(p) = \tan[(p-1/2)\pi]$$

Then, (Eq-3) gives:

$$\lim_{e\to 0}\frac{\tan[(e-0.5)\pi]-\tan[(2e-0.5)\pi]}{\tan[(2e-0.5)\pi]-\tan[(4e-0.5)\pi]} =$$

$$\lim_{e\to 0}\frac{-1/(2e)}{-1/(4e)} 2^1, \gamma = -1$$

$$a_n = w(F) = F^{-1}(1) = 0$$

and $b_n = \tan[\pi(0.5-n^{-1})]$

which shows that the Cauchy distribution does not belong to domain of max-Weibull distribution and we write $F \notin D_{\max}(W_M)$. But belongs to the Frechet minimal domain of attraction. A possible selection of the constants, according to (Eq-4), is:

$a_n = w(F) = F^{-1}(1) = 0$ and $b_n = \tan[\pi(0.5 - n^{-1})]$

Some examples of distributions which belong to the domain attraction of max-Weibull distributions are given in [10,12].

4 Model Estimation

There are several methods to estimate parameters of max-Weibull. We focus on maximum likelihood estimation (MLE) because of nice asymptotic. [18], it has described how a max-Weibull can be fitted with MLE in,.[10]. The MLE is based on maximizing the likelihood of the observed sample. The likelihood function of this random sample is the joint density of the n random variables and is a function of the unknown parameter. Thus Consider the Weibull pdf given in (1) and letting $\mu = 0$, we have, then likelihood function will be:

$$L = \prod_{i=1}^{n} f(x_i, \gamma, \sigma) = \prod_{i=1}^{n} (\frac{\gamma}{\sigma})(\frac{x_i}{\sigma})^{\gamma-1} e^{-(\frac{x_i}{\sigma})^{\gamma}} \quad 7$$

On taking the logarithms of (6), differentiating with respect to γ and σ in turn and equating to zero, we obtain the estimating equations:

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \ln x_i - \frac{1}{\sigma} \sum_{i=1}^{n} x_i^{\gamma} \ln x_i = 0$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma} \sum_{i=1}^{n} x_i^{\gamma} = 0 \quad 8$$

On eliminating σ between these two equations and simplifying, we have:

$$\frac{\sum_{i=1}^{n} x_i^{\gamma} \ln x_i}{\sum_{i=1}^{n} x_i^{\gamma}} - \frac{1}{\gamma} - \frac{1}{n} \sum_{i=1}^{n} x_i^{\gamma} = 0 \quad 9$$

which may be solved to get the estimate of $\hat{\mu} = \gamma$. This can be accomplished by the use of standard iterative procedures (i.e., Newton-Raphson method). Once γ is determined, σ can be estimated using equation (8) as:

$$\sigma = \frac{\sum_{i=1}^{n} x_i^{\gamma}}{n} \quad 10$$

For computational, see for more details, [19,20] .

5 Model Diagnostics

The reason for fitting a statistical model to data is to make conclusions about some aspect of the population from which the data were drawn.

5.1 Selecting Models by Graphical Methods

In this section, three important graphical plots to select a suitable to make conclusions about some aspect of the population from which the data were drawn. Now, we will focus on some graphical methods for checking whether a fitted model is in agreement with the data such that, probability paper (P-P) plot, Quantile Quantile (Q-Q) plot and Return level (R-L) plot by [18], which are often more informative for our purposes and that deals with the problem of selection models by diagnostic plots. Moreover, many popular estimation methods from EVT turn out to be directly based on these graphical tools. Firstly, we describe the P-P Plot.

5.1.1 The P-P Plot

The P-P Plot can be used to distinguish visually between different distribution. Let $x_1, x_2, ..., x_n$ be a sample from a given population with estimated cdf $\hat{F}(x_i)$. The scatter plot points of P-P plot is given by:

$$(\hat{F}(x_i), \frac{i}{n+1}, i = 1,2,...n) \quad 11$$

Where $P_{i:n}$ is called plotting positions (PP) is defined by :

$$P_{i,n} = \frac{i-\alpha}{n+\beta}, i = 1,2,..., n \quad 12$$

Here we use $\alpha = 0$ and $\beta = 1$ that is, other alternative can be found in [10,12].

If $\hat{F}$ is a reasonable· model for the population distribution, the points of the P-P plot should lie close to the unit diagonal. Substantial departures from linearity provide evidence of a failure in $\hat{F}$ as a model for the data.

5.1.2 The Q-Q Plots

The quantile-quantile plot (Q-Q plot) is a convenient visual tool to examine whether a sample comes from a specific distribution. Let $\hat{F}$ be an estimate of F based on $x_1, x_2,..., x_n$. The scatter plot of the points are given by:

$$(\hat{F}^{-1}(\frac{i}{n+1}), x_i, i = 1,2,...n) \quad 13$$

is called a Q-Q plot, thus, the Q-Q plot show the estimated versus the observed quantiles. If the model fits the data well, the pattern of points on the Q-Q plot will exhibit a straight line and the picture strongly suggests that the data follow a distribution is acceptable. The P-P plot and the Q-Q plot contain the same information expressed on a different scale.

5.1.3 Return Level Plot (RLP)

From the fitted **max-Weibull** distribution, we can estimate how often the extreme quantiles occur with a certain return level (RT). The return value is defined as a value that is expected to be equalled or exceeded on average once every interval of time (T) (with a probability of 1/T). In [8,9], the RL can be calculated by solving this equation (i.e., by inverting **Eq-2** in max-Weibull). Quantile max - Weibull is given by:

$$X_P = \mu - \sigma(-Logp)^{1/\gamma} \quad 14$$

We estimate above parameters by using MLE and substituting the parameters γ, μ, σ by their estimates $\hat{\gamma}, \hat{\mu}, \hat{\sigma}$ to get $\hat{x}_p$. The RLP represents the points (x_P, y_P) where $x_P = -\log(\mu - x)$ and $y_P = -\log(-Logp)$ then confidence intervals are usually added to this plot to increase its information. Consequently, on domain of attraction condition of max-Weibull distributions appear as concave.

5.2 Selecting Models by Hypothesis Testing

The procedures described in the previous section are based on graphical displays. Another approach for selecting problem is to formulate the problem into a hypothesis testing framework. For testing the data come from a d.f. of **max-weibull** defined in (1), the hypothesis testing can be written as:

$$H_0 : \gamma = 0 \ versus \quad H_1 : \gamma < 0 \ (Weibull \quad 15$$

So under H_0, the likelihood ratio test (LRT) is defined as:

$$LRT = -2\log(\frac{L_0}{L_1}) \approx \chi^2_{1,1-\alpha} \quad 16$$

where L_0 and L_1 are the value of the log likelihood under the null and alternative hypothesis respectively. Under H_0 the LRT is with a χ^2_1 distribution, The H_0 will be rejected at significance α -level if $LRT > \chi^2_{1,1-\alpha}$ or P-value less than 0.05 ($P-value < 0.05$). For more details see [16]. Here, we are particularly interested in the case $\gamma < 0$, motivated by the conclusion drawn in the graphical preliminary analysis. Some recent references for tests for selection of extreme value models,[17].

6 Application on Flood data

The data sets used in this study consist of Annual Maximum (AM), to climate data of max-Temperature in Sebha over the period 1981 to 2020 from consist of 40 year blocks. The first period of the data from 1981-200 consist of 20 year blocks and the second period from 2001-2020 consist of 20 year blocks. Data analysis of this study is carried out using R -programming with packages of fExtreme and ismev.

7 Results and Discussion

The first step for data analysis is to see the graphical behaviour and present descriptive statistical summary for max-Temperature data. The annual block maxima of three periods are displayed in **Fig.1**.

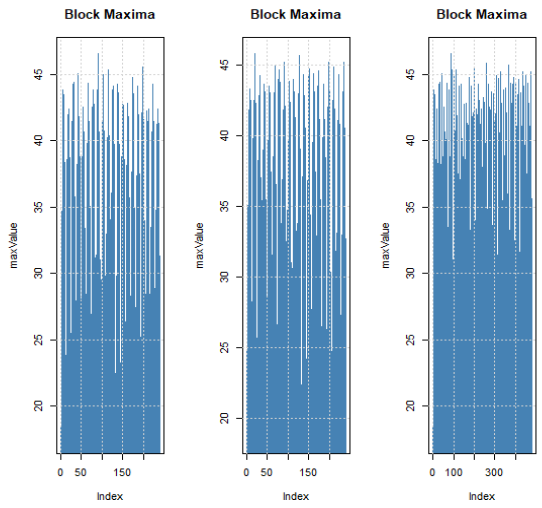

Fig.1: Left panel: plots the yearly block maxima from 1981-2000. Mid panel from 2001-2020 and right panel from 1981-2020.

7.1 Summary Statistics

Firstly, we present some results based on descriptive statistics of max-temperature application data and the results are reported in Table-1.

Table 1: Summary of the main descriptive statistic

Statistics	1981-200	2001-2020	1981-2020
Min	17.58	18.61	17.58
Max	46.62	45.87	46.62
Mean	35.11	35.98	35.54
Median	37.60	38.40	38.02
Skewnes	**-0.52**	**-0.56**	**-0.54**

In Table 1, it can be seen that the results of all descriptive statistics at max-Temperature. The skewness are negative (here, $sk < 0$) for three period, that mean the distribution has left tail ($\gamma < 0$), so there is a good reasons to think that the distribution of these data an appropriate by using max-Weibull modeling fitting. We apply MLE to estimate the three parameters of distribution for the application data. The AM data is fitted by MLE to get the point estimate of shape, location and scale and summarizes the results in Table-2.

Table 2: Parameters estimation of max-Temperature by MLE, and Standard errors are in parentheses

Methods	Parameters	Period of data		
		1981-2000	2001-2020	1981-2020
MLE	Shape	-0.63 (0.03)	-0.75 (0.02)	-0.65 (0.01)
	Location	33.76 (0.55)	35.13 (0.53)	34.30 (0.37)
	Scale	8.23 (0.44)	8.17 (0.41)	8.13 (0.25)
Tail behaviour		$\gamma < 0$	$\gamma < 0$	$\gamma < 0$
Test	LRT	796.73	778.56	1581.01
	P-value	0.00	0.00	0.00

The results listed in Table 2, show the estimates of three parameters. The point estimates of shape for three periods of data are negative and indicate that the distribution of data has left tail and **max-**Weibull distribution is an appropriate for these data of max-Temperature. Also the P-value of the LR test of each period is smaller than all the significance levels (P-value<0.01) and it also shows that the max-Weibull is good for model fitting for these data. This is confirmed by the standard diagnostic graphical checks in **Fig-2-4**

7.2 Diagnostic Plots

In order to get an idea about the tail behaviour of the distribution. We present in **Fig 2-4**, the various diagnostic plots for assessing the accuracy of the model fitted for an application data of max-Temperature are shown in four different plots; P-P-plot, Q-Q-plot, Return level plot and density plot.

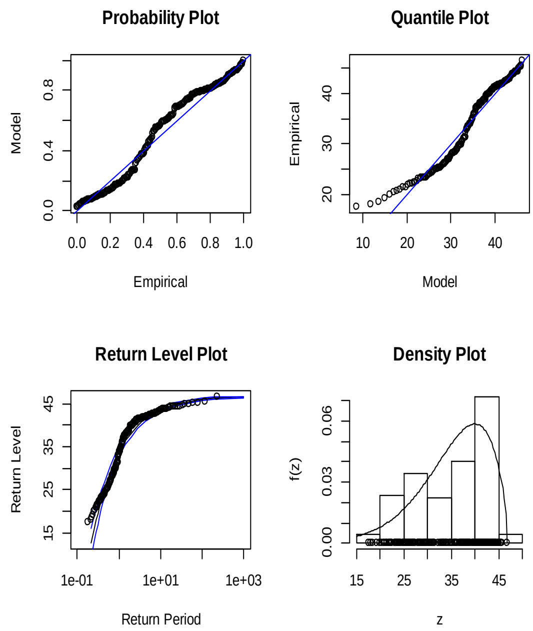

Fig. 2: Four different plots; P-P, Q-Q, R-L and density plot of first period (1981-200).

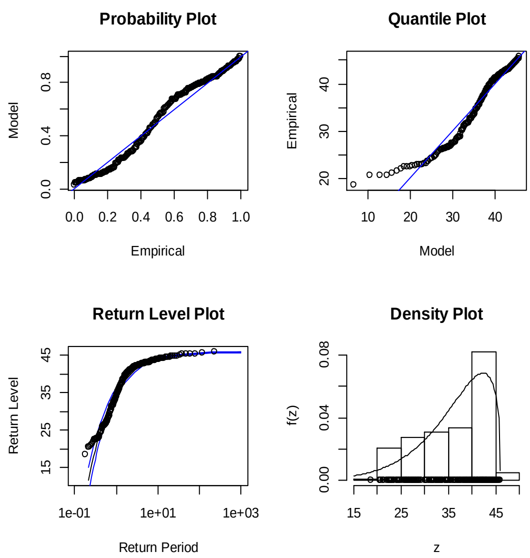

Fig. 3: Four different plots; P-P, Q-Q, R-L and density plot data of second period(2001-2020).

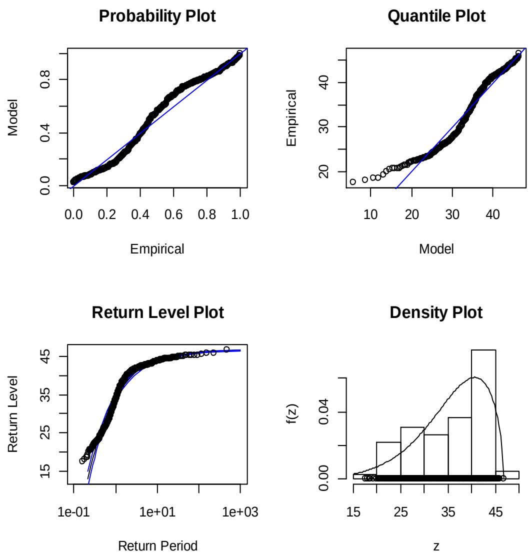

Fig. 4: Four different plots; P-P, Q-Q, R-L and density plot data of total period(1981-2020).

The various diagnostic plots for assessing the accuracy of the max-Weibull model fitted to the Port Pirie data are shown in **Fig. 2-4**. The graph of the P-P and the Q-Q plot are shown the data came from a max-Weibull distribution when each set of plotted points is near-linear. the R-L plot suggest that the model departures of the negative shape estimate ($\gamma < 0$). Finally, the corresponding density estimate seems consistent with the histogram of the data. Consequently, all four diagnostic plots lend support to max-Weibull model are the best fit.

8 Conclusion

In this paper we focused on the extreme max-temperature data and illustrated how extreme value theory can be used to model extreme temperature. This paper investigates some theoretical and practical aspects of the use max-Weibull distribution. This distribution is one of the probability distributions used to model extreme events. Firstly, In this paper, we discuss the problem which is called the Max-domain of attraction condition for belonging to the domain of max-Weibull distribution by using Gnedenko's, von Mises, necessary and sufficient condition and practical aspects of the use max-Weibull distribution. Secondly, diagnostic plots, the Q-Q Plot, P-P Plot, R-L Plot and density plot for the fitted max-Weibull distribution are shown. Consequently, all diagnostic plots lend support to the fitted max-Weibull model. Third, Also the P-value of the LR test of each period is smaller than all the significance levels (P-value<0.01) and it also shows that the max-Weibull is good for model fitting for these data. Some issues that would be considered for future works in this study are the problem of min-domain of attraction condition of min-Weibull distribution and other some diagnostics tests in order to select the best fit model. Finally, all results of problem in the domain of attraction condition of max-Weibull distribution (under linear) will try reformulated under power and we look forward to see other application to recognize.

نمذجة القيم العظمي باستخدام توزيع ويبل مع التطبيق

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الملخص: يظهر توزيع ويبل في العديد من التطبيقات العملية عند ظهور القيم المتطرفة. في هذه الورقة، الهدف الأول هو مناقشة مشكلة شروط مجال جذب توزيعات القيم العظمي لتنتمي إلى مجال توزيع ويبل للقيم العظمي باستخدام شرطي كلا من كاستيلو وديهان. الهدف الثاني هو استخدام الرسم البياني واختبار الفروض للكشف عن التوزيع المناسب. والهدف الثالث في هذه الدراسة هو نمذجة سلوك بيانات درجات الحرارة العظمي باستخدام توزيع ويبل للقيم العظمي. تم استخدام طريقة الإمكان الأعظم لتقدير معلمات التوزيع وأظهرت النتائج أنه في جميع الحالات الثلاثة أن نموذج ويبل للقيم العظمي مناسب بالاعتماد علي اختبار نسبة الإمكان والرسم البياني. حيث تم استخدام برنامج R في العمليات الحسابية لتقدير المعالم وإجراء الاختبار و الرسومات البيانية التشخيصية.

الكلمات المفتاحية: توزيع ويبل للقيم العظمي، ثوابت الاتزان، رسمة الاحتمال، رسمة التجزئة، رسمة مستوي العائد، شروط مجال الجاذبية ،طريقة الإمكان الأعظم لتقدير المعالم.

References

[1] Fisher, Ronald Aylmer, and Leonard Henry Caleb Tippett. "Limiting forms of the frequency distribution of the largest or smallest member of a sample." In Mathematical Proceedings of the Cambridge Philosophical Society Cambridge University Press, vol. 24, no. 2, (1928) : PP180-190.

[2] R. Von Mises, “Über allgemeine Quadraturformeln.,” J. für die reine und Angew. Math., vol. 174, pp. 56–67, 1936.

[3] B. Gnedenko, “Sur la distribution limite du terme maximum d'une serie aleatoire,” Ann. Math., pp. 423–453, 1943.

[4] De Haan, Laurens. "Weak limits of sample range." Journal of Applied Probability 11.4 (1974): 836-841.

[5] Reiss, Rolf-Dieter, Michael Thomas, and R. D. Reiss. Statistical analysis of extreme values. Vol. 2. Basel: Birkhäuser, 1997.

[6] Reiss, Thomas. "Drug discovery of the future: the implications of the human genome project." Trends in biotechnology 19.12 (2001): 496-499.

[7] Kotz, Samuel, and Saralees Nadarajah. Extreme value distributions: theory and applications. World Scientific, ISBN 1860942245, (2000):P1.

[8] Coles, Stuart, Joanna Bawa, Lesley Trenner, and Pat Dorazio. An introduction to statistical modeling of extreme values. Vol. 208. London: Springer, ISBN 1852334592, (2001):PP4-12,59-64.

[9] Beirlant, Jan, Yuri Goegebeur, Johan Segers, and Jozef Teugels. Statistics of extremes: theory and applications. John Wiley & Sons, ISBN 0-471-97647-4, (2004):PP19-42.

[10] E. Castillo, A. S. Hadi, N. Balakrishnan, and J.-M. Sarabia, Extreme value and related models with applications in engineering and science. Wiley Hoboken, NJ, 2005:P108,P148

[11] Reiss, Henning, and Ingrid Kröncke. "Seasonal variability of benthic indices: an approach to test the applicability of different indices for ecosystem quality assessment." Marine Pollution Bulletin 50.12 (2005): 1490-1499.

[12] J. Beirlant, Y. Goegebeur, J. Segers, and J. L. Teugels, *Statistics of extremes: theory and applications*. John Wiley & Sons, 2006:PP56-59.

[13] Matas, Jiri, Charles Galambos, and Josef Kittler. "Robust detection of lines using the progressive probabilistic hough transform." Computer vision and image understanding 78.1 (2000): 119-137.

[14] Galambos, Janos, and Nicholas Macri. "The life length of humans does not have a limit." Journal of Applied Statistical Science 9.4 (2000): 253-264.

[15] Nadarajah, Saralees. "The exponentiated Gumbel distribution with climate application." Environmetrics: The official journal of the International Environmetrics Society 17.1 (2006): 13-23.

[16] K. Abbas and T. Yincai, “Comparison of Estimation Methods for Frechet Distribution with Known Shape.,” *Casp. J. Appl. Sci. Res.*, vol. 1, no. 10, 2012.

[17] E. Omey, F. Mallor, and E. Nualart, “An introduction to statistical modelling of extreme values. Application to calculate extreme wind speeds,” 2009.

[18] Bardsley, Earl. "The Weibull distribution as an extreme value model for transformed annual maxima." Journal of Hydrology (New Zealand) 58.2 (2019): 123-131.

[19] Wuertz, Diethelm. "fextremes: Rmetrics-extreme financial market data." R package version 2100 (2013).

[20] Gilleland, Eric, Mathieu Ribatet, and Alec G. Stephenson. "A software review for extreme value analysis." Extremes 16

(2013): 103-119.

[21] Ikbal, N. A. M., Halim, S. A., & Ali, N. (2022). Estimating Weibull Parameters Using Maximum Likelihood Estimation and Ordinary Least Squares: Simulation Study and Application on Meteorological Data. Math. Stat, 10, 269-292.

[22] Gilleland, E., Ribatet, M., & Stephenson, A. G. A software review for extreme value analysis. Extremes, 16, 103-119. (2013).