CYCLIC WEAKLY AMENABLE SEMIGROUP ALGEBRAS WHICH ARE NOT WEAKLY AMENABLE

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Abstract

In this paper we shall construct some examples where the semigroup algebra $\ell^1(S)$ is cyclic weakly amenable but not weakly amenable ,where S is a commutative, 0-cancellative, $nil^{\sharp}$-semigroup.

I. INTRODUCTION

We follow [1] to recall some definitions and some preliminaries. Let $\mathcal{A}$ be a Banach algebra, and let X be a *Banach* $\mathcal{A}$*-bimodule*. A linear map $D: \mathcal{A} \to X$ is a *derivation* if it satisfies the equation:

$$D(ab) = a \cdot Db + Da \cdot b \quad (a, b \in \mathcal{A}) .$$

In this paper we shall only consider bounded derivations. Given $x \in X$ and define the map $\delta_x: \mathcal{A} \to X$ by the equation:

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These derivations are *inner* derivations.

Let X^* be the *dual space* of X. Then X^* is a Banach $\mathcal{A}$-bimodule with respect to the operations given by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle \quad and$$
$$\langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (a \in \mathcal{A}, x \in X, \lambda \in X^*)$$

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation D from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule X^* is inner, for each Banach $\mathcal{A}$-bimodule X.

A Banach algebra $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule over itself. Then a Banach algebra $\mathcal{A}$ is *weakly amenable* if every bounded derivation $D: \mathcal{A} \to \mathcal{A}^*$ is inner.

A linear map $T: \mathcal{A} \to \mathcal{A}^*$ is cyclic if $T(a_1)(a_0) = (-1)T(a_0)(a_1)$ for all $a_0, a_1 \in \mathcal{A}$; in other words, $\langle a_0, T(a_1)\rangle + \langle a_1, T(a_0)\rangle = 0 \quad (a_0, a_1 \in \mathcal{A})$.

In particular, $\langle a, T(a)\rangle = 0 \quad (a \in \mathcal{A})$.

The space of all bounded, cyclic derivations from $\mathcal{A}$ to $\mathcal{A}^*$ is denoted by $\mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*)$, and the set of all cyclic inner derivations from $\mathcal{A}$ to $\mathcal{A}^*$ is denoted by $\mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*)$. It can be seen that every inner derivation is cyclic, and so $\mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*) = \mathcal{N}^1(\mathcal{A}, \mathcal{A}^*)$. The *first-order cyclic cohomology group* is defined by

$$\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) = \frac{\mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*)}{\mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*)} = \mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*)/\mathcal{N}^1(\mathcal{A}, \mathcal{A}^*) .$$

A Banach algebra $\mathcal{A}$ is *cyclic weakly amenable* if $\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$.

It is obvious that $\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) \subseteq \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*)$, so each weakly amenable Banach algebra is cyclic weakly amenable.

Let S be a non-empty set, and let s be an element of S. The characteristic function of $\{s\}$ is denoted by δ_s. We define the *Banach space*

$$\ell^1(S) := \left\{ f : S \to \mathbb{C},\ \ f = \sum_{s\in S} \alpha_s \delta_s , \sum_{s\in S} |\alpha_s| < \infty \right\},$$

where $\|f\| = \sum_{s\in S} |\alpha_s| < \infty$.

The *dual space* of $\mathcal{A} = \ell^1(S)$ is $\mathcal{A}^* = \ell^\infty(S)$, where

$$\ell^\infty(S) = \left\{ f : S \to \mathbb{C},\ \ \|f\| = \sup_{s\in S} |f(s)| < \infty \right\},$$

with the duality given by:

$$\langle f, \lambda \rangle = \sum_{s\in S} f(s)\lambda(s) \quad (f \in \ell^1(S), \lambda \in \ell^\infty(S)).$$

Let S be a semigroup. Then the *convolution product* of two elements f and g in the Banach space $\ell^1(S)$ is defined by the formula:

$$f * g = \left(\sum_{s\in S} \alpha_s \delta_s \right) * \left(\sum_{t\in S} \beta_t \delta_t \right) = \sum \left\{ \left(\sum_{st=r\in S} \alpha_s \beta_t \right) \delta_r \right\}$$

The inner sum will vanish if there are no s and t such that $st = r$.

Clearly, $(\ell^1(S), *)$ is a Banach algebra; it is called the *semigroup algebra* of S.

We shall need to use the following remark:

Remark 1.1

Let S be a semigroup, and take g to be a function on $S \times S$. For $a, b \in S$, define

$$T_g(\delta_a, \delta_b) = g(a, b),$$

and then extend T_g to be a bilinear function on $\ell_0^1(S) \times \ell_0^1(S)$. In the case where g is bounded by M, T_g extends to a bounded, bilinear functional on $\ell^1(S) \times \ell^1(S)$.

Explicitly,

$$\left| T_g \left(\sum_i \alpha_i \delta_{a_i}, \sum_j \beta_j \delta_{b_j} \right) \right| = \left| \sum_{i,j} \alpha_i \beta_j g(a_i, b_j) \right| \le M \sum_i |\alpha_i| \sum_j |\beta_j|.$$

Now define $\widetilde{T_g} : \ell^1(S) \to \ell^\infty(S)$ by

$$\langle h, \widetilde{T_g}(f) \rangle = T_g(f, h) \quad (f, h \in \ell^1(S)).$$

Then $\widetilde{T_g}$ is a bounded linear map and

$$\langle \delta_b, \widetilde{T_g}(\delta_a) \rangle = g(a, b) \quad (a, b \in S).$$

Throughout the paper, S denotes a countable commutative $nil^\sharp$-semigroup which is the unitization of a nil semigroup S^- (that is, a semigroup S^- with zero such that for all $x \in S^-$, there is an $n \in \mathbb{N}$ such that $x^n = o$), and which is zero-cancellative (that is, for all $a, b, c \in S$, $ab = ac \ne o$ implies $b = c$).

Following [2], we shall write $V_S(x)$ for the set of divisors of x in a unital semigroup S, that is,

$$V_S(x) = \{ y \in S : \exists z \in S, yz = x \}.$$

The set $V_S^*(x)$ is the collection of all functions $g : V_S(x) \to \mathbb{C}$ satisfying the logarithmic condition

$$g(ab) = g(a) + g(b) \quad (ab \in V_S(x)) \qquad \text{(I.1)}$$

In our paper, we shall apply the characterization of cyclic weak amenability of some certain commutative semigroup algebras $\ell^1(S)$, as established in [3], where S is a commutative, 0-cancellative, $nil^\sharp$-semigroup, as introduced in [2] where S is a finite or an infinite semigroup. Then we shall construct some examples where $\ell^1(S)$ is cyclic weakly amenable but not weakly amenable.

II Finite semigroups

In this section we shall discuss cyclic weak amenability of the semigroup algebra $\mathcal{A} = \ell^1(S)$, where S is a finite, commutative, 0-cancellative, $nil^\sharp$-semigroup.

An element $a \ne e \in S$ is an *atom* if $V_S(a) = \{o, e\}$. we shall show that $\mathcal{A}$ is cyclic weakly amenable only when S has exactly one atom, or $S = \{o, e\}$.

Proposition II.1

Let S be a finite, commutative, 0-cancellative, $nil^{\sharp}$-semigroup with just one atom. Then the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is cyclic weakly amenable.

Proof

Suppose that S is a finite, commutative, 0-cancellative, $nil^{\sharp}$-semigroup with just one atom a. Then the semigroup S can be written as

$T_n = \{e, a, a^2, \dots, a^{n-1}, a^n = o\}$

for some $n \in \mathbb{N}$ with $n \geq 2$. We suppose that $\mathcal{A}_n = \ell^{1}(T_n)$

Take a derivation $D: \mathcal{A}_n \to \mathcal{A}_n^*$. Then D is bounded because the semigroup algebra $\mathcal{A}_n$ is finite dimensional.

We have $D(\delta_e) = D(\delta_o) = 0$. It can be proved that

$D(\delta_a) = \lambda_e \delta_e^* + \lambda_1 \delta_a^* + \cdots + \lambda_{n-2} \delta_{a^{n-2}}^*$

for some $\lambda_e, \lambda_1, \dots, \lambda_{n-2} \in \mathbb{C}$.

We know that D is cyclic if and only if satisfies the equation:

$$\langle f, D(g)\rangle + \langle g, D(f)\rangle = 0 \quad (f, g \in \mathcal{A}). \quad \text{(II.1)}$$

Tak

$f = \delta_{a^k}$ and $g = \delta_a$ for $k = 0, \dots, n-1$, where $a^0 = e$ and $\delta_{a^0} = \delta_e$. Then

$$\langle f, D(g)\rangle + \langle g, D(f)\rangle = \langle \delta_{a^k}, D(\delta_a)\rangle + \langle \delta_a, D(\delta_{a^k})\rangle$$

$$= \langle \delta_{a^k}, D(\delta_a)\rangle + \langle \delta_a, k\delta_{a^{k-1}} D(\delta_a)\rangle$$

$$= \langle \delta_{a^k}, D(\delta_a)\rangle + k\langle \delta_a, \delta_{a^{k-1}} D(\delta_a)\rangle$$

$$= (k+1)\langle \delta_{a^k}, D(\delta_a)\rangle,$$

and so, by (II.1), we have

$$\langle \delta_{a^k}, \lambda_e \delta_e^* + \lambda_1 \delta_a^* + \cdots + \lambda_k \delta_{a^k}^* + \cdots + \lambda_{n-2} \delta_{a^{n-2}}^* \rangle = 0,$$

hence $\lambda_k = 0$ for all $k = 0, \dots, n-2$. So $D = 0$. Therefore $\mathcal{HC}^{1}(\mathcal{A}, \mathcal{A}^*) = \{0\}$.

Thus the proposition is proved. □

Proposition II.2

Let S be a finite, commutative, 0-cancellative, $nil^{\sharp}$-semigroup with exactly two atoms a and b. Then the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is not cyclic weakly amenable.

Proof

Suppose that the elements a and b are two atoms of the semigroup S. We have to discuss two cases:

1. Let $ab \neq o$. We *claim* that $V_S(ab) = \{e, a, b, ab\}$. For if $u|ab$ then either $u = e$ or there is an atom v such that $v|u$. If $a|u$, then $u = au_1|ab$ implies that $u_1|b$, so $u_1 = e$ or $u_1 = b$. Hence ($u = e$ or a or ab). Similarly, if $b|u$, we have $u = b$ or $u = ab$.

Now we define the function $g \in V_S^*(ab)$ by:

$$g(x) = \begin{cases} 0 & if \;\; x = e \;\; or \;\; x = ab \\ 1 & if \;\; x = a \\ -1 & if \;\; x = b \end{cases} \quad \text{(II.2)}$$

By using (II.2), it is clear that $g(xy) = g(x) + g(y)$ for all $x, y \in S$ such that $xy|ab$. Thus by [3, Proposition 2.1], the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is not cyclic weakly amenable.

2. Let $ab = o$. Then the semigroup S can be defined as

$$S = \{e, a, a^2, \dots, a^{n-1}, a^n = o = b^m, b, b^2, \dots b^{m-1}, ab = o\},$$

for some $n, m \geq 2$.

A non-zero bounded sensible function $\varphi: C_{(a,b)} \to \mathbb{C}$ can be defined on the class

$$C_{(a,b)} = \{(a,b), (b,a)\}$$

by:

$$\varphi(x,y) = \begin{cases} 1 & if \; x = a \;\; and \;\; y = b \\ -1 & if \; x = b \;\; and \;\; y = a, \end{cases} \quad \text{(II.3)}$$

So that by [3, Theorem 2.4], the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is not cyclic weakly amenable.

Thus the proposition is proved. □

III. Infinite semigroups

In this section we shall discuss cyclic weak amenability of the semigroup algebra $\mathcal{A} = \ell^{1}(S)$, where S is an infinite, commutative, 0-cancellative, $nil^{\sharp}$-semigroup. we shall show that $\mathcal{A}$ is not cyclic weakly amenable if S has at least two distinct atoms.

Proposition III.1

Let S be a commutative, 0-cancellative, $nil^{\sharp}$ -semigroup. Suppose that S has atoms a, b such that $ab = o$. Then the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is not cyclic weakly amenable.

Proof

We shall show that there is a non-zero bounded sensible function φ on the equivalence class $C_{(a,b)} \in M_o^-$ with the equivalence relation defined on the set M_o^- as in [3, Definition 1.3].

We *claim* that

$C_{(a,b)} = \{(a,b),(b,a)\}$. (III.1)

If so, a non-zero bounded sensible function $\varphi: C_{(a,b)} \to \mathbb{C}$ can be defined by:

$$\varphi(x,y) = \begin{cases} 1 & if \;\; x = a \; and \; y = b \\ -1 & if \;\; x = b \; and \; y = a \,, \end{cases}$$

so that by [3, Theorem 2.4], the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is not cyclic weakly amenable.

To prove (III.1), suppose that $(a,b) \sim (p,q)$, so that either we write $a = \alpha\beta$, and $(\alpha\beta, b) \sim (\alpha, \beta b)$. Otherwise we write $b = \gamma\lambda$, and $(a, \gamma\lambda) \sim (a\gamma, \lambda)$. But a is an atom, so if $a = \alpha\beta$ then $\alpha = e$ and $\beta = a$ otherwise if $\alpha = a$ and $\beta = e$ then the new pair is (a,b) . Also if $\alpha = e$ and $\beta = a$ then the new pair is $(e, ab) = (e, 0) \notin M_o^-$.

Similarly, since b is an atom, we cannot get a new pair out of $(b\gamma, \lambda)$ with $\gamma\lambda = b$. Thus the proposition is proved. □

Theorem III.2

Let S be a commutative, 0-cancellative, $nil^{\sharp}$ -semigroup. Suppose that S has at least two distinct atoms. Then the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is not cyclic weakly amenable.

Proof

Suppose that S has two distinct atoms a, b.

We have two cases. If $ab = o$, then, by Proposition (III.1), we have that $\mathcal{A} = \ell^{1}(S)$ is not cyclic weakly amenable.

Suppose that $ab \neq o$. We claim that there is a non-zero bounded function g in $V_S^*(ab)$ with $g(ab) = 0$. We define g by

$$g(x) = \begin{cases} 1 & if \;\; x = a \\ -1 & if \;\; x = b \\ 0 & otherwise\,, \end{cases} \quad \text{(III.2)}$$

and we must show that $g(xy) = g(x) + g(y)$ for all $x, y \in V_S(ab)$.

Suppose that $x, y \in V_S(ab)$. We have $g(x) = g(y) = g(xy) = 0$ unless one of x, y, xy is a or b. If $xy = a$ or $xy = b$ then the pair $\{x, y\}$ is $\{e, a\}$ or $\{e, b\}$ and $g(x) + g(y) = g(xy)$ hence we may assume towards a contradiction that $x = a$.

We *claim* that $y = e$ or $y = b$ with $g(x) + g(y) = g(xy)$.

For if y is not e or b we have $ay = xy | ab$ so $y | b$ and b is not an atom. But this is a contradiction. Thus there is $g \in V_S^*(ab)$ with $g(ab) = 0$. Therefore by [3, Theorem 2.4], the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is not cyclic weakly amenable. □

IV. Examples

We shall now establish some nice examples for some infinite semigroups, where the semigroup algebra $\ell^{1}(S)$ is cyclic weakly amenable but not weakly amenable.

Example IV.1

Let S be the semigroup $([0,1], \oplus)$ where

$a \oplus b = \min(a+b, 1) \quad (a, b \in S)$.

Then S is an infinite, commutative, 0-cancellative $nil^{\sharp}$ -semigroup, and the semigroup algebra $\mathcal{A} = \ell^{1}(S)$ is cyclic weakly amenable but not weakly amenable.

Note that, in S , the unit element is 0 and the zero element is 1 .

Let $0 < p < 1$. To show that $\mathcal{A}$ is not weakly amenable take the function g to be the identity function, so that $g(x) = x$ for all $x \in V_S(p)$. We have $V_S(p) = [0, p]$, hence $st \in V_S(p)$ implies that $s + t \leq p$, so that

$g(s \oplus t) = s + t = g(s) + g(t)$.

Thus $g \neq 0 \in V_S^*(p)$. By [2, Theorem 1.1], we see that $\mathcal{A}$ is not weakly amenable.

To prove cyclic weak amenability, we have to show that every element h of $V_S^*(p)$ is a multiple of the function g, and then show that there are no bounded sensible functions as before. Since $g(p) \neq 0$, this shows that there is no non-zero element $h \in V_S^*(p)$ with $h(p) = 0$. Let $\alpha = h(p)$. Then the log-condition show that $h(p/n) = \alpha/n$ for all $n \in \mathbb{N}$ and $h(kp/n) = \alpha k/n$ for $k, n \in \mathbb{N}$ and $0 < k \leq n$. We claim that $h(rp) = r\alpha$ for every $0 \leq r \leq 1$ otherwise the function $[0,1] \to \mathbb{R}^+$, $r \mapsto h(rp)$ is discontinuous. Suppose that ρ is irrational with $0 < \rho < 1$ and $h(\rho p) = \rho\beta$ for $\beta \neq \alpha$. Then $h(r\rho p) = r\rho\beta$ for $r \in \mathbb{Q}$ with $r\rho < 1$.

There are rationals $r_n < \rho$ with $r_n \to \rho$ when $n \to \infty$, so that

$$h(\rho p) - h(r_n p) = \rho\beta - r_n\alpha \to \rho(\beta - \alpha) \neq 0 \quad as \quad n \to \infty.$$

Fix $M > 0$. For large n, we have

$$h(\rho p) - h(r_n p) > M(\rho - r_n) > 0.$$

But by the log-condition we have

$$h((\rho - r_n)p) = h(\rho p) - h(r_n p) > M(\rho - r_n).$$

Take $k \in \mathbb{N}$ with $\frac{1}{2} < k(\rho - r_n) \leq 1$, so that we have ,

$$h(k(\rho - r_n p)) = k(h(\rho p) - h(r_n p)) > Mk(\rho - r_n) > \frac{1}{2}M,$$

so that the function h is not bounded. But this is a contradiction, because elements of $V_S^*(p)$ must be bounded. Thus $\dim V_S^*(p) = 1$ for $0 < p < 1$ and there is no non-zero function $h \in V_S^*(p)$ with $h(p) = 0$. The function $g_p \in V_S^*(p)$ with $g_p(p) = 1$ is $g_p(x) = x/p$ for all $x \in V_S(p)$.

Now we shall prove that there is no non-zero, bounded, sensible function φ on any equivalence class $C_{(a,b)}$. We first claim that

$$C_{(a,b)} = \{(\alpha, \beta) : \alpha + \beta = a + b\},$$

with $a + b \geq 1$.

For say $\alpha < a$, we have $a = \alpha \oplus (a - \alpha)$ so that

$$(\alpha \oplus (a - \alpha), b) \sim (\alpha, b + a - \alpha).$$

Similarly for $\beta < b$. But the relation $\sim$ cannot relate pairs (a, b) and (c, d) with $a + b \neq c + d$. So take $\alpha_1, \alpha_2, \beta \in S$ such that $\alpha = \alpha_1 + \alpha_2$ and $\alpha + \beta = a + b$ so that we have

$$\varphi(\alpha_1, \beta + \alpha_2) = g_\alpha(\alpha_1) \cdot \varphi(\alpha, \beta) = \frac{\alpha_1}{\alpha} \cdot \varphi(\alpha, \beta), \text{(IV.1)}$$

on the other hand we have

$$\varphi(\alpha, \beta) = -\varphi(\beta, \alpha) = -\varphi(\beta, \alpha_1 + \alpha_2 = -g_{\alpha_2+\beta}(\beta) \cdot \varphi(\alpha_2 + \beta, \alpha_1)$$

$$= \beta/(\alpha_2 + \beta) \cdot \varphi(\alpha_1, \alpha_2 + \beta). \quad \text{(IV.2)}$$

If φ is non-zero, choose α, β such that $\varphi(\alpha, \beta) \neq 0$ and by comparing (IV.1) and (IV.2) we must have $\beta/(\alpha_2 + \beta) = \alpha/\alpha_1$ for every

$\alpha = \alpha_1 + \alpha_2$; but the equation $\beta/(\alpha - \alpha_1 + \beta) = \alpha/\alpha_1$ is not true for all values of α_1. Therefore, there is no non-zero bounded sensible function φ, and by [3, Proposition 2.3], the semigroup algebra $\mathcal{A} = \ell^1(S)$ is cyclic weakly amenable.

Example IV.2

Let $A \subset \mathbb{C}$ *be the subset* $\{0\} \cup \{a + \imath b \in \mathbb{C} : 0 < a < 1\}$. *Suppose that* $S = A \cup \{\theta\}$ *such that*

$$s \oplus \theta = \theta \oplus s = \theta \quad (s$$

and for each $z, w \in A$ we have

$$z \oplus w = \begin{cases} z + w & if \ z + w \in A \\ \theta & if \ z + w \notin A. \end{cases} \quad \text{(IV.3)}$$

Indeed, the identity of S is 0 and the zero element is θ. Then we claim that S is an infinite, commutative, 0-cancellative $nil^\sharp$-semigroup, and the semigroup algebra $\mathcal{A} = \ell^1(S)$ is cyclic weakly amenable but not weakly amenable.

To show that $\mathcal{A}$ is not weakly amenable take $z \in A$ and define the function g to be the real-part function, so that $g(w) = \Re(w)$ for all $w \in V_S(z)$. We have $V_S(z) = \{w \in S : z \in wS\} = \{a + \imath\, b : 0 < a < \Re(z)\}$,

(so that the identity function is not bounded on $V_S(z)$). For $u, v \in A$ with $\Re(u) + \Re(v) < 1$ we have $\theta \neq uv \in V_S(z)$ and $g(u \oplus v) = \Re(u) + \Re(v) = g(u) + g(v)$.

Thus $g \neq 0 \in V_S^*(z)$. By [2, Theorem 1.1] we see that $\mathcal{A}$ is not weakly amenable.

Since the identity function is not bounded on the set A, for $z \in A$ the linear space $V_S^*(z)$ does not contain the identity function. We claim that $V_S^*(z)$ is all complex multiples of the function $g(w) = \Re(w) \quad (w \in A)$.

For if we have $w \in V_S(z)$, so that $\Re(w) < \Re(z)$ or $w = z$. We claim that for $g \in V_S^*(z)$ we have

$$g(w) = g(w + \imath\, \lambda) \quad (w \in V_S(z)).$$

Now for all $w \in V_S(z)$, and for some $\lambda \in \mathbb{R}$, we see that

$$g(\tfrac{w}{2} + \imath\, \tfrac{\lambda}{2}) + g(\tfrac{w}{2} - \imath\, \tfrac{\lambda}{2}) = g(w),$$

also

$$2g(\tfrac{w}{2} + \imath\, \tfrac{\lambda}{2}) = g(w + \imath\, \lambda) \quad and \quad 2g(\tfrac{w}{2} - \imath\, \tfrac{\lambda}{2}) = g(w - \imath\, \lambda),$$

so that

$$g(w) = \tfrac{1}{2}(g(w + \imath\, \lambda) + g(w - \imath\, \lambda)),$$

or

$$g(w + \imath\, \lambda) = \tfrac{1}{2}(g(w + 2\imath\, \lambda) + g(w)),$$

hence

$$g(w + \imath\, 2\lambda) = 2g(w + \imath\, \lambda) - g(w).$$

Thus

$$g(w + \imath\, 3\lambda) = 2g(w + 2\imath\, \lambda) - g(w + \imath\, \lambda) = 3g(w + \imath\, \lambda) - 2g(w),$$

and similarly for $n \in \mathbb{N}$, we have

$$g(w + \imath\, n\lambda) = ng(w + \imath\, \lambda) - (n-1)g(w),$$

so that

$$g(w + \imath\, n\lambda) - g(w) = n(g(w + \imath\, \lambda) - g(w)).$$

By induction, for $n \in \mathbb{N}$ we see that

$$g(w + \imath\, n\lambda) = g(w) + n(g(w + \imath\, \lambda) - g(w))$$

for all $w \in V_S(z)$,

so we have $g(w + \imath\, \lambda) = g(w)$ (otherwise g is not bounded). Also we claim that $g(\frac{mz}{n}) = \frac{m}{n} g(z)$ for all $m \leq n$ in $\mathbb{N}$; once again we must have $g(\alpha z) = \alpha g(z)$ for all $0 < \alpha < 1$ otherwise g will not be bounded. Then if $k = \frac{g(z)}{\Re(z)} \in \mathbb{C}$, we have for $0 < \alpha < 1$,

$$g(\alpha z + \imath\, \lambda) = g(\alpha z) = \alpha g(z) = \alpha k \Re(z) = k\Re(\alpha z + \imath\, \lambda).$$

Thus $V_S^*(z)$ consists of multiples of the function $g(w) = \Re(w)$ for each $w \in V_S(z)$ and $g_z(w) = \frac{\Re(w)}{\Re(z)}$.

To prove that $\mathcal{A}$ is cyclic weakly amenable we seek to show that there is no non-zero bounded sensible function φ on any equivalence class $C_{(a,b)}$.

We *claim* that

$$C_{(a,b)} = \{(\alpha, \beta) : \alpha, \beta \in A \quad and \quad \alpha + \beta = a + b\}.$$

For once cannot have $(a, b) \sim (a', b')$ without $a + b = a' + b'$. Conversely, if $(\alpha, \beta) \in M_o^-$ with $a + b = \alpha + \beta$ we *claim* that $(\alpha, \beta) \in C_{(a,b)}$.

Given $\epsilon > 0$ with $\Re(b) + \epsilon < 1$, and $(a, b) \sim (\Re(a) - \epsilon, \Re(b) + \epsilon)$. Certainly, $(a, b) \sim (b, a)$ so that $C_{(a,b)} = C_{(a-\epsilon, b+\epsilon)}$, so we can assume that $\Re(a) \neq \Re(\alpha)$. If $\Re(a) > \Re(\alpha)$ then $\alpha | a$ in S , $a = \alpha + (a - \alpha)$ for $a, a - \alpha \in A$ so that $(a, b) \sim (\alpha, b + a - \alpha) = (\alpha, \beta)$.

Similarly, if $\Re(a) < \Re(\alpha)$ we have $(a, b) \sim (\alpha, \beta)$ also.

Suppose that φ is a sensible function, so that $\varphi(\frac{a+b}{2}, \frac{a+b}{2}) = 0$. If $c + d = a + b$ then either we have $\Re(c) \leq \Re(\frac{a+b}{2})$ or $\Re(d) \leq \Re(\frac{a+b}{2})$, so we may assume that $\Re(c) \leq \Re(\frac{a+b}{2})$ and $\Re(\frac{c}{3}) < \Re(\frac{a+b}{2})$, so $\frac{c}{3} \mid \frac{a+b}{2}$ in S, and

$$\varphi(c,d) = 3\varphi(\frac{c}{3}, d + \frac{2c}{3}) =$$

$$3g_{\frac{a+b}{2}}(\frac{c}{3})\varphi(\frac{a+b}{2}, \frac{a+b}{2}) = 0\ .$$

Thus the semigroup algebra $\mathcal{A} = \ell^1(S)$ is cyclic weakly amenable but not weakly amenable.

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