A Comparison of the Pre-test and Shrinkage Estimators for a Finite Population Mean in a Bivariate Normal Distribution with Equal Marginal Variances Under Distrust Coefficient

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Abstract— The estimation of the mean of a bivariate normal population with unknown variance is considered in this paper when uncertain non-sample prior information on the value of the mean and a coefficient of distrust on the null hypothesis is available. Alternative estimators are defined to incorporate both the sample as well as the non-sample information in the estimation process. Some of the important statistical properties of the restricted, preliminary test, and shrinkage estimators are investigated.

The performances of the estimators are compared based on the criteria of unbiasedness and mean square error in order to search for a "best" estimator. Both analytical and graphical methods are explored. There is no superior estimator that uniformly dominates the others.

In addition, the results showed that neither the preliminary test estimator nor the shrinkage estimator dominates one another except for large dimensions. However, if the non-sample information regarding the value of the mean is close to its true value, the shrinkage estimator over performs the rest of the estimators.

***Keywords*: Uncertain non-sample prior information; maximum likelihood, restricted, preliminary test shrinkage estimators.**

I. INTRODUCTION

The classical estimators of unknown parameters are based completely on the sample data, and ignore any other kind of non-sample prior information. However, it is a natural expectation that the quality of the estimators may improve if non sample prior information is incorporated in the estimation of the parameters. Any such estimators that combine both sample and non-sample prior information are likely to perform better that the exclusive sample based estimator under specific statistical criterion. A number of estimators have been introduced in the literature that, under particular situation, over performs the classical exclusive sample based estimators when judged by criteria such as the mean square error and square error loss function (Khan and Saleh, 2001).

There have been many studies in the area of improved estimators following the seminal work of Bancroft (1944). later Han and Bancroft (1968). They developed the preliminary test estimator that uses uncertain non-sample prior information in addition to sample information. Stein (1956) introduced the Stein rule (shrinkage) estimator for multivariate normal population that dominates the usual maximum likelihood estimator (MLE) under quadratic loss function.

In a series of papers Saleh and Sen (1978, 1985) explored the preliminary test approach to Stein-rule estimation under nonparametric set up. Many authors have contributed to this area, notably Judge and Bock (1978), Stein (1981), Matta and Casella (1990),Khan (1998), Later Khan and Saleh (1995, 1997) investigated

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the problem for a family of Student-t populations. Recently, Khan, S and Saleh, A.K.Md.E. (2001), on the comparison of pre-test and Shrinkage estimator for the univariate normal mean, have used the coefficient of distrust $0 \le d \le 1$, a measure of degree of lack of trust on the null hypothesis, in the estimation of parameters. This coefficient of distrust reflects on the reliability of the prior information. Xianchao (2016) developed shrinkage estimation in family of distributions with quadratic variance function based on two specific cases: the location-scale family and the natural exponential families. It was showed that shrinkage estimator was asymptotically optimal in its own class and superior performance compared to the classical empirical Bayes and many other competing shrinkage estimators.

Later on, Mathenge (2019) considered two shrinkage estimators of rates based on Bayesian methods to estimate the mean of the multivariate normal distribution in when the variance was unknown using the chi-square random variable. Was found that the limits of the maximum likelihood estimator based on the two risk ratios forms obtained when n and p tend to infinity.

Recently, Hamdaoui et al (2020) considered two forms of shrinkage estimators of the mean of a multivariate Normal random variable in the Bayesian case when the variance is unknown, are showed that the Bayes and shrinkage estimators were minimax n and p were finite.

This paper mainly focuses on the estimation of the finite population mean, from a bivariate normal distribution with equal marginal variances with uncertain non-sample prior information on the value of the mean; this can be expressed into the null hypothesis, which may be true but not sure. In section 2.0, we consider the estimation of the means of bivariate normal distribution and consider a new shrinkage estimation of means following Saleh (2006) in the absence of Stein-type estimators since the dimension<3. The three alternative estimators are defined in section 3.0. The expressions of bias and mean square error (MSE) function of the estimators are obtained in section 4.0. Comparative studies of the relative efficiency of the estimators are included in section 5.0. In this papers, we consider the problem of estimation of the mean μ_1 of one of the components of a bivariate normal distribution with equal marginal variances from a sample of size n. The result of a preliminary test of hypothesis that the mean μ_1 and μ_2 of the two components of the bivariate normal distribution are equal is used to define an estimator for μ_1. The bias and mean square error of this estimator are studied and the regions in the parameter space in which the estimator has smaller mean square error, the sample mean of the first component, are determined. The efficiency of this estimator relative to the usual estimator is tabulated and the tables can be used to determine a proper choice of significance level of the preliminary test.

In this process, we define three biased estimators: the restricted estimator (RE) with a coefficient of distrust, the preliminary test estimator (PTE) as a linear combination of the usual estimator and the RE, and the shrinkage estimator (SE) by using the preliminary test approach. We investigate the bias and the mean square error function, both analytically and graphically to compare the performance of the estimators.

The relative efficiency of the estimators is also studied to search for a better choice. Extensive computation has been used to produce graphs and tables to critically check various effects on the properties of the estimators.

2.0 The Model and Some preliminaries

Let $(X_{1i}, X_{2i})(i = 1,2,3,\ldots,n)$ be a random sample of size n from a bivariate normal distribution with parameters $\mu_1,\mu_2,\sigma_1^2,\sigma_2^2$ and ρ . The parameter ρ is introduced to take into account the fact that most multistage interpenetrating samples have a series of common primary, and secondary, sampling units.

Restricting ourselves to the case $\sigma_1^2 = \sigma_2^2 = \sigma^2$, where σ^2 is unknown parameter and we are interested in the estimation of μ_1 when it is a priori suspected that $\mu_1 = \mu_2$ may be true.

Also, assume that uncertain non-sample prior information on the value of μ_1 is available, either from previous study or from practical experience of the researchers or experts. Let the non-sample prior information be expressed in the form of null hypothesis

$$H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_A : \mu_1 \ne \mu_2 \tag{2.1}$$

Which may be true, but not sure? Then using the test statistic, as well as the sample and non-sample information to define the preliminary test and shrinkage estimators of the unknown μ_1 it is well known that the MLE of the μ_1 is unbiased. We wish to search for an alternative estimator of the mean that is biased but may well have some superior statistical property in terms of the mean square error. We are primarily concerned with the estimation of μ_1 , when it is suspected but not sure that $\mu_1 = \mu_2$, i.e., with uncertain prior information about μ_1 . The unrestricted estimator (UE) of μ_1 is the usual (MLE) estimator given by

$$\tilde{\mu}_1 = \frac{1}{n}\sum_{j=1}^{n} x_{1j.} \tag{2.2}$$

Hence, the bias and the mean square error (MSE) of $\tilde{\mu}_1$ are given by:

$$B_1(\tilde{\mu}_1) = E[\tilde{\mu}_1 - \mu_1] = 0. \tag{2.3}$$

It is well known that the sampling distribution of MLE of $\tilde{\mu}_1$ is normal with mean μ_1 and variance equal to $\frac{\sigma^2}{n}$.

$$M_1(\tilde{\mu}_1) = E[\tilde{\mu}_1 - \mu_1]^2 = \frac{\sigma^2}{n} \tag{2.4}$$

3.0 Alternative Estimators

As part of incorporating the uncertain non-sample prior information into the estimation process, first we combine the exclusive sample based estimator, $\tilde{\mu}_1$ with

$$\hat{\mu}_1(d) = d\tilde{\mu}_1 + (1-d)\hat{\mu}_1^0, \quad 0 \le d \le 1,$$

Where $\hat{\mu}_1{}^o$ is the estimator under H_0 in (2.1) given by

$$\hat{\mu}_1^0 = \frac{\tilde{\mu}_1 + \tilde{\mu}_2}{2}, \quad \tilde{\mu}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij}; \quad i = 1, 2.$$

This estimator of μ_1 is called the restricted estimator (RE), where d is the degree of distrust on the null hypothesis, H_0. Now, $d = 0$ means there is no distrust in the H_0 and we get $\hat{\mu}_1(d=0) = \hat{\mu}_1^0$; while $d = 1$ means there is complete distrust in the H_0 and we get $\hat{\mu}_1(d=1) = \tilde{\mu}_1$. If 0 < d < 1, the degree of distrust is an intermediate value

$$\hat{\mu}_1^{PTE}(d) = \tilde{\mu}_1 - (1-d)(\tilde{\mu}_1 - \hat{\mu}_1^0)I(|t_v| \le t_{\frac{\alpha}{2}}),$$

Where

$$t_v = (\tilde{\mu}_1 - \tilde{\mu}_2)\left[\frac{s_{11} + s_{22} - 2s_{12}}{n(n-1)}\right]^{-\frac{1}{2}}$$

t_v is test statistic for testing the null-hypothesis H_0 and I (A) is the indicator function of the set A and $t_{\alpha/2}$ is the critical value chosen for the two-side α level test based on the Student-t distribution with $v = (n-1)$ degrees of.

$$\hat{\mu}_1^{PTE}(d) = \tilde{\mu}_1 - (1-d)(\tilde{\mu}_1 - \hat{\mu}_1^0)I(F \le F_\alpha)$$

Where F_α is the $(1-\alpha)th$ quartile of a central $F-distribution$ with $(1, v)$ degrees of freedom. The (PTE) is an extreme choice between $\hat{\mu}_1(d)$ and $\tilde{\mu}_1$. Hence, it does not allow any smooth transition between the two

$$\hat{\mu}_1^{SE} = \hat{\mu}_1^0 + (\tilde{\mu}_1 - \hat{\mu}_1^0)\left\{1 - c|t_v|^{-1}\right\}$$

, where C is shrinkage constant. Now, if $|t_v| = \frac{|(\tilde{\mu}_2 - \tilde{\mu}_1)\sqrt{n(n-1)}|}{\sqrt{S_{11} + S_{22} - 2S_{12}}}$ is large, $\hat{\mu}_1^{SE}$ tends towards $\tilde{\mu}_1$, while for small $|t_v|$ equaling c, $\hat{\mu}_1^{SE}$ tends towards $\hat{\mu}_1^0$ similar to the preliminary test estimator. The shrinkage estimator dose not depends on the level of significance, unlike the preliminary test estimator.

4.0 Some Statistical Properties

The mean square error function of the restricted estimator $\hat{\mu}_1(d)$ is

$$M_2(\hat{\mu}_1(d)) = \frac{\sigma^2}{n} - \frac{\sigma^2}{2n}(1-d^2)(1-\rho) + \frac{\sigma^2}{2n}\Delta^2(1-d)^2(1-\rho).$$

And $\Delta^2 = \frac{n(\mu_2 - \mu_1)^2}{2\sigma^2(1-\rho)}.$

the non-sample prior information presented in the form of a null hypothesis defined in (2.1) . First, consider a simple linear combination of $\hat{\mu}_1{}^o$ and $\tilde{\mu}_1$ as:

(3.1)

:

(3.2)

which results in an interpolated value between $\hat{\mu}_1^0$ and $\tilde{\mu}_1$ given by (3.1). We may rewrite the above estimator in (3.1) as:

$$\hat{\mu}_1(d) = \tilde{\mu}_1 - (1-d)(\tilde{\mu}_1 - \hat{\mu}_1^0) \quad (3.3)$$

Following Saleh (2006), we define the preliminary test estimator (PTE) of μ_1 define as:

(3.4)

Further

$$s_{ij} = \sum_{k=1}^{n}(x_{ik} - \tilde{\mu}_i)(x_{jk} - \tilde{\mu}_j), \tilde{\mu}_i = \frac{1}{n}\sum_{k=1}^{n} x_{ik} \; ; (i = 1, 2; j = 1, 2)$$

The bias and the mean square error have been derived by Ahsanullah (1971) for odd of sample size.

We may rewrite the above equation (3.4) as :

(3.5)

extreme values. Also, it depends on the pre-selected level of significance of the test.

To overcome these problems, we consider the shrinkage estimator (SE) of μ_1 defined as follows:

(3.6)

In this section, we derive the bias and the mean square error (MSE) functions of the estimators in (3.3), (3.5) and (3.6). Also, we discuss some of important features of these estimators.

4.1 The bias and MSE of the RE

The bias function of the restricted estimator (RE), $\hat{\mu}_1(d)$ is

$$B_2(\hat{\mu}_1(d)) = \frac{(1-d)}{2}(\mu_2 - \mu_1). \quad (4.1)$$

(4.2)

Where Δ^2 is the departure constant from the null-hypothesis. The value of this constant is 0 when the null hypothesis is true; otherwise it is always positive.

The performance of the estimators change with the change in value of Δ.

4.2 The bias and MSE of the PTE

$$B_3\left(\hat{\mu}_1^{PTE}(d)\right)=\frac{1}{2}(1-d)(\mu_2-\mu_1)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right). \tag{4.3}$$

Where $H_{q_1,q_2}(.,\Delta^2)$ is the cumulative distribution of a non-central $F-$ distribution with (q_1,q_2) degrees of freedoms and non-centrality parameter Δ^2 .This bias function of PTE depends on the coefficient of distrust and Δ^2.

To evaluate the expression in (4.3) we used the following theorem :(see Saleh (2006))

Theorem 4.1.If $Z\sim N(\Delta;1)$ and $\Phi(Z^2)$ is a Borel measurable function, then

$$E\left\{Z\Phi(Z^2)\right\}=\Delta E\Phi\left(\chi_3^2(\Delta^2)\right).$$

$$M_3\left(\hat{\mu}_1^{PTE}(d)\right)=\frac{\sigma^2}{n}-(1-d^2)\frac{\sigma^2}{2n}(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)$$
$$+(1-d)\frac{\sigma^2}{2n}(1-\rho)\Delta^2\left[2H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)-(1+d)H_{5,v}\left(\frac{1}{5}F_{1,v}(\alpha),\Delta^2\right)\right] \tag{4.4}$$

Where $Z=\frac{(\tilde{\mu}_2-\tilde{\mu}_1)}{\sigma}\sqrt{\frac{n}{2(1-\rho)}}$ is distributed as $N(\Delta,1)$,where $\Delta^2=\frac{n(\mu_2-\mu_1)^2}{2\sigma^2(1-\rho)}$.

4.2.1 Some Properties of MSE of PTE

$$\frac{\sigma^2}{n}\left[1-\frac{1}{2}(1-d^2)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\right]<\frac{\sigma^2}{n}. \tag{4.5}$$

Thus, at $\Delta^2=o,$ the PTE of μ_1 performs better than $\tilde{\mu}_1$, the UE. As $\alpha\to 0,$

$$\frac{\sigma^2}{n}\left[1-\frac{1}{2}(1-d^2)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\right]\to\frac{\sigma^2(1+d^2)(1-\rho)}{2n}. \tag{4.6}$$

Which is the MSE of $\hat{\mu}_1(d)$. On the other hand, if

$$F_\alpha\to 0 \quad \text{And} \quad H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\to 0$$

Which is the MSE of $\tilde{\mu}_1$.

(ii) As $\Delta^2\to\infty,$ $H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\to 0,$ this means the expression at (4.4) tends towards $\frac{\sigma^2}{n}$ the MSE of the UE.

(iii) Since $H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)$ is always greater than $H_{5,v}\left(\frac{1}{5}F_{1,v}(\alpha),\Delta^2\right)$ for any value of α replacing $H_{5,v}\left(\frac{1}{5}F_{1,v}(\alpha),\Delta^2\right)$ by $H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)$, the expression in (4.4) becomes

On the other hand, the expression in (4.4) may be rewritten as

The bias function of the preliminary test estimator (PTE), $\hat{\mu}_1^{PTE}(d)$ is

Furthermore, to obtain the mean square error of $\hat{\mu}_1^{PTE}(d)$ we need the following theorem:

Theorem 3.2 if $Z\sim N(\Delta;1)$ and $\Phi(Z^2)$ is a Borel measurable function, then

$$E\left(Z^2\Phi(Z^2)\right)=E\left[\Phi\left(\chi_3^2(\Delta^2)\right)\right]+\Delta^2E\left[\Phi\left(\chi_5^2(\Delta^2)\right)\right].$$

The proof of the above two theorems (4.1) and (4.2) are given in Appendix B2 of Judge and Bock (1978).

The MSE of the preliminary test estimator (PTE), $\hat{\mu}_1^{PTE}(d)$ is

(i) Under the null hypothesis $\Delta^2=o,$ and hence the MSE of the PTE $\hat{\mu}_1^{PTE}$ equals

$$H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\to 1$$

Then,

Then

$$\frac{\sigma^2}{n}\left[1-\frac{1}{2}(1-d^2)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\right]\to\frac{\sigma^2}{n}. \tag{4.7}$$

$$\geq\frac{\sigma^2}{n}\left[1+\frac{1}{2}(1-d^2)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)\left[(1-d)\Delta^2-(1+d)\right]\right],$$

$$\geq\frac{\sigma^2}{n}$$

Whenever

$$\Delta^2>\frac{1+d}{1-d}. \tag{4.8}$$

$$\frac{\sigma^2}{n}\left[1+\frac{1}{2}(1-d)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)\left\{2\Delta^2-(1+d)\right\}-\left(1-d^2\right)(1-\rho)H_{5,v}\left(\frac{1}{5}F_{1,v}(\alpha),\Delta^2\right)\right]$$

$$\leq\frac{\sigma^2}{n}$$ Whenever

$$\Delta^2<\frac{1+d}{1-d} \qquad (4.9)$$

This means that the MSE of PTE $\hat{\mu}_1^{PTE}(d)$ as a function of Δ² crosses the constant line $M_1(\tilde{\mu}_1)=\frac{\sigma^2}{n}$ in the interval $\left(\frac{1+d}{2},\frac{1+d}{1-d}\right)$.

Figures (4.1) display the behavior of the relative efficiency function of the PTE for different values of α with the change in the value of Δ². The graphs illustrate the different features for values of $d=0$ and $d=0.5$ when $\rho=-0.7$,and n=5.

Clearly the (MSE and hence the) relative efficiency of the preliminary test estimator compared with the unrestricted estimator depends on level of significance α of the test of null-hypothesis and the departure parameter Δ²

Let the relative efficiency of the PTE with respect to the UE be denoted by $E_3\left(\alpha;\Delta^2\right)$ Which is given by

$$E_3\left(\alpha;\Delta^2\right)=RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right)$$

4.2.2 Determination of optimum α for the PTE

$$RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right)=\frac{MSE\left(\tilde{\mu}_1\right)}{MSE\left(\hat{\mu}_1^{PTE}\right)}=\left[1+g\left(\Delta^2\right)\right]^{-1}, \qquad (3.18)$$

Where

$$g\left(\Delta^2\right)=\frac{1}{2}(1-d)\Delta^2(1-\rho)\left[2H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)-(1+d)H_{5,v}\left(\frac{1}{5}F_{1,v}(\alpha),\Delta^2\right)\right]$$

$$g\left(\Delta^2\right)=-\frac{1}{2}\left(1-d^2\right)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right) \qquad (3.19)$$

The efficiency function attains its maximum at Δ² =0 for all α given by

$$E_3(\alpha;0,\rho)=\left[1-\frac{1}{2}\left(1-d^2\right)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\right]^{-1}\geq 1. \qquad (4.12)$$

As Δ² departs from the origin, $E_3\left(\alpha;\Delta^2,\rho\right)$ decreases monotonically crossing the line $E_3\left(\alpha;\Delta^2,\rho\right)=1$ to a minimum at Δ²= Δ_{min} , with minimum efficiency equal to E0, then from that

A point on increases monotonically towards 1 as $\Delta\to\infty$ from below. In order to choose an optimum level of significance with maximum relative efficiency we adopt the following rule: if it is known that $0\leq\Delta\leq\frac{1+d}{1-d}$, $\hat{\mu}_1$ is always chosen since $E_3\left(0;\Delta^2,\rho\right)$ is maximum for all Δ² in this interval. Generally, Δ² is unknown, we consider two cases.

(i) Suppose the experimenter does not know the size of α but knows $\rho=\rho_0$ and wants to accept an estimator which has relative efficiency not less than E_0 . Then among the of estimator with $\alpha\in A_\alpha$, where

$$A_\alpha=\left\{\alpha:\backslash\quad E\left(\alpha;\Delta^2,\rho_0\right)\geq E_0\quad For\ all\ \Delta^2\right\}. \qquad (4.13)$$

An estimator $\hat{\mu}_1^{PTE}(d)$ is chosen which maximizes $E_3\left(\alpha;\Delta^2,\rho_0\right)$ over all α and Δ². Thus, we solve the following equation for α $Max_\alpha Min_{\Delta^2}E\left(\alpha;\Delta^2,\rho_0\right)=E_0$.

The solution α^* provides a maximum rule for the optimum level of significance of the preliminary test. For practitioners, Tables (4.1) and (4.2) provide the maximum $\left(E^*\right)$ and minimum $\left(E_0\right)$ relative efficiency of the PTE, the values of Δ_0 at which the minimum relative efficiency occur and the intersection point between PTE and UE $\left(\Delta_1\right)$ for selected values of α when d=0 , n=5 and n=12.

From Table (4.2), as an example of selecting an optimal level of significance, if one wishes to have a guaranteed minimum relative efficiency of $E_0^{min}=0.1959$ of the PTE with a sample size of n=12, ρ=0.30 and d=0 he has to select a level of significance, $\alpha^* = 0.35$ with maximum relative efficiency he can obtain is 1.0702.

(ii) Suppose the experimenter does not know the size of Δ^2 and ρ, but wants an estimator which has a relative efficiency not less than E_0, then he has to look for α^* for which $E(\alpha;\Delta^2,\rho) \ge E_0$ for all Δ^2 and ρ. Tables (4.1) and (4.2) can be used for finding α^*. Suppose n=5 and n=15, d=0.25 and ρ∈A, where

A={ρ:ρ = -0.9, -0.7, -0.1, 0.1, 0.3, 0.7, 0.9}

And the experimenter wants an estimator of relative efficiency not less 0.567.

From Table (4.2), we fine that by choosing $\alpha^* = 0.2$ with maximum relative efficiency is 1.0195. Also, Tables (4.1) and (4.2) show the range of Δ for which $\hat{\mu}_1^{PTE}$ dominates $\tilde{\mu}_1$ for selected values of α and d for example using Table (4.2), if n=12, ρ=0.10 and d=0 with α=0.05 and 0 ≤ Δ ≤ 1.7632, then PTE dominates UE but outside the interval the UE dominates the PTE.

4.3 The bias and MSE of SE

The bias function of the Shrinkage estimator (SE) $\hat{\mu}_1^{SE}$ is

$$B_4\left(\mu_1^{SE}\right) = -cE\left(\tilde{\mu}_1 - \hat{\mu}_1^0\right)\left|t^{-1}\right|$$

$$B_4\left(\hat{\mu}_1^{SE}\right) = \frac{c}{2}E\left(\tilde{\mu}_2 - \tilde{\mu}_1\right)\left|t^{-1}\right|$$

Where

$$|t| = |\tilde{\mu}_1 - \tilde{\mu}_2|\left[\frac{S_{11}+S_{22}-2S_{12}}{n(n-1)}\right]^{-\frac{1}{2}}$$

$$B_4\left(\hat{\mu}_1^{SE}\right) = -\frac{c}{\sqrt{2n}}E\left(\frac{(\tilde{\mu}_1-\tilde{\mu}_2)\sqrt{n/2(1-\rho)}}{\left|(\tilde{\mu}_1-\tilde{\mu}_2)\sqrt{n/2(1-\rho)}\right|}\right)$$

$$B_4\left(\hat{\mu}_1^{SE}\right) = -\frac{c}{\sqrt{n}}\sqrt{\frac{1}{2}}E\left[\frac{Z}{|Z|}\right] \qquad (4.14)$$

Where $Z = \frac{(\tilde{\mu}_2 - \tilde{\mu}_1)}{\sigma}\sqrt{\frac{n}{2(1-\rho)}}$ Is distributed as $N(\Delta,1)$, where

$$\Delta = \sqrt{\frac{n}{2(1-\rho)}}\left(\frac{\mu_2-\mu_1}{\sigma}\right)$$

Now, we use the following theorem to evaluate(4.14)

Theorem 4.3. If $Z \sim N(\Delta,1)$ and $\Phi(Z^2)$ is a Borel measurable function, then

$$E\left[\frac{Z}{|Z|}\right] = 1 - 2\Phi(-\Delta)$$

Where $\Phi(.)$ is the c.d.f. of the standard normal variable.

Now, using Theorem (3.3) we find

$$B_4\left(\hat{\mu}_1^{SE}\right) = \frac{c^2}{n}\frac{1}{2}K^2\left[2\Phi(\Delta)-1\right]^2, \qquad (4.15)$$

Where, $K = \sqrt{\frac{2}{n-1}}\frac{\Gamma(n/2)}{\Gamma(n-1/2)}$.

Furthermore, to obtain the mean square error of $\hat{\mu}_1^{SE}$ we need the following theorem (see Saleh (2006)).

Theorem 4.4. If $Z \sim N(\Delta,1)$; then

$$E(|Z|) = \sqrt{\frac{2}{\pi}}e^{-\Delta^2/2} + \Delta\left\{2\Phi(\Delta)-1\right\},$$

The proof of the above two theorems (4.3) and (4.4) are given in Khan and Saleh (2001).The MSE of the Shrinkage estimator (SE) $\hat{\mu}_1^{SE}$ is

$$M_4\left(\hat{\mu}_1^{SE}\right) = \frac{\sigma^2}{n}\left[1+\frac{1}{2}c^2(1-\rho) - c(1-\rho)K\sqrt{\frac{2}{\pi}}e^{\frac{-\Delta^2}{2}}\right]. \qquad (4.16)$$

The value of c which minimizes (4.16) depends on Δ^2 and is given by

$$C^* = K\sqrt{\frac{2}{\pi}}e^{-\frac{\Delta^2}{2}} \qquad (4.17)$$

To make c^* independent of Δ^2, we choose $c^0 = K\sqrt{\frac{2}{\pi}}$. thus, optimum reduces to

$$M_4\left(\mu_1^{SE}\right) = \frac{\sigma^2}{n}\left[1+\frac{K^2}{\pi}(1-\rho)\left(1-2e^{-\frac{\Delta^2}{2}}\right)\right]. \qquad (4.18)$$

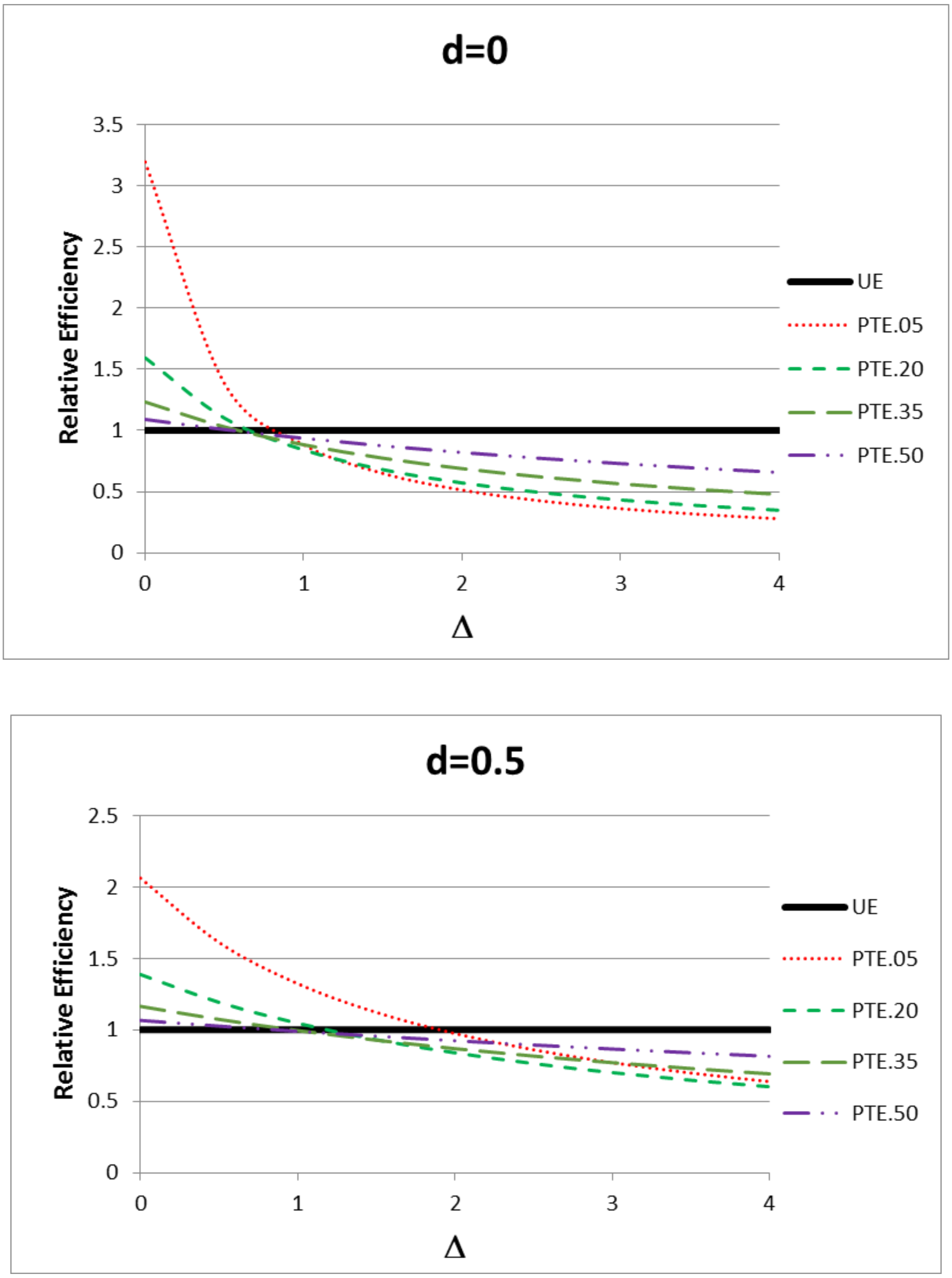


Figure 4.1: Graph of the Relative Efficiency of PTE for selected values of d and α for n=5 and ρ=-0.7

Table4.1: Values of the maximum and minimum efficiency of *PTE* recommended significance levels and the intersection point between *PTE* and *UE* (Δ_α) for $d = 0$, $n = 5$

α/ρ		-0.9	-0.7	-0.1	0.1	0.3	0.5	0.9
0.0 5	E^*	4.3021	3.1924	1.7998	1.5713	1.3943	1.2531	1.0421
	E_0	0.0305	0.0339	0.0515	0.0622	0.0786	0.1066	0.3738
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.8383	0.8383	0.8383	0.8383	0.8383	0.8383	0.8383
0.1	E^*	2.6828	2.2791	1.5702	1.4227	1.3006	1.1977	1.0341
	E_0	0.0335	0.0372	0.0564	0.0681	0.0859	0.1162	0.3968
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.7565	0.7565	0.7565	0.7565	0.7565	0.7565	0.7565
0.2	E^*	1.7125	1.593	1.3173	1.2455	1.181	1.123	1.0224
	E_0	0.0436	0.0485	0.073	0.0878	0.1101	0.1476	0.4641
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.6623	0.6623	0.6623	0.6623	0.6623	0.6623	0.6623
0.25	E^*	1.5046	1.4287	1.2409	1.1889	1.141	1.0968	1.018
	E_0	0.0511	0.0568	0.0851	0.1021	0.1276	0.17	0.5059
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.6316	0.6316	0.6316	0.6316	0.6316	0.6316	0.6316
0.35	E^*	1.2669	1.2323	1.1389	1.1109	1.0841	1.0587	1.0112
	E_0	0.0738	0.0817	0.1209	0.144	0.1778	0.2324	0.6021
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.5872	0.5872	0.5872	0.5872	0.5872	0.5872	0.5872
0.45	E^*	1.1418	1.125	1.0775	1.0625	1.048	1.0338	1.0066
	E_0	0.1135	0.1252	0.1811	0.2128	0.2579	0.3273	0.7087
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.5569	0.5569	0.5569	0.5569	0.5569	0.5569	0.5569
0.5	E^*	1.1017	1.09	1.0564	1.0457	1.0352	1.0249	1.0049
	E_0	0.1444	0.1586	0.2256	0.2626	0.3141	0.3906	0.7622
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.5454	0.5454	0.5454	0.5454	0.5454	0.5454	0.5454

Table 4.2: Values of the maximum and minimum efficiency of *PTE* recommended significance levels and the intersection point between *PTE* and *UE* (Δ_α) for $d=0$, $n=12$

α/ρ		-0.9	-0.7	-0.1	0.1	0.3	0.5	0.9
0.05	E^*	3.57	2.8098	1.7146	1.5174	1.361	1.2337	1.0394
	E_0	0.0295	0.0329	0.0499	0.0604	0.0763	0.1037	0.3664
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.7632	0.7632	0.7632	0.7632	0.7632	0.7632	0.7632
0.1	E^*	2.3283	2.0427	1.4932	1.3703	1.2661	1.1767	1.031
	E_0	0.0334	0.0372	0.0564	0.068	0.0858	0.1161	0.3965
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.6886	0.6886	0.6886	0.6886	0.6886	0.6886	0.6886
0.2	E^*	1.5726	1.4832	1.2671	1.2084	1.1549	1.106	1.0195
	E_0	0.046	0.0511	0.0769	0.0924	0.1157	0.1549	0.4781
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.614	0.614	0.614	0.614	0.614	0.614	0.614
0.25	E^*	1.4075	1.3496	1.2014	1.1589	1.1194	1.0825	1.0155
	E_0	0.0552	0.0613	0.0916	0.1097	0.1368	0.1816	0.5259
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.5914	0.5914	0.5914	0.5914	0.5914	0.5914	0.5914
0.35	E^*	1.2167	1.1896	1.115	1.0921	1.0702	1.0492	1.0095
	E_0	0.0824	0.0912	0.1342	0.1593	0.1959	0.2543	0.6303
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.5597	0.5597	0.5597	0.5597	0.5597	0.5597	0.5597
0.45	E^*	1.1154	1.102	1.0637	1.0515	1.0396	1.028	1.0055
	E_0	0.1294	0.1425	0.2043	0.2389	0.2875	0.361	0.7386
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.5387	0.5387	0.5387	0.5387	0.5387	0.5387	0.5387
0.5	E^*	1.0827	1.0733	1.0463	1.0375	1.029	1.0205	1.004
	E_0	0.1655	0.1814	0.2551	0.2951	0.3499	0.4297	0.7903
	Δ_0	40.1	40.1	40.1	40.1	40.1	40.1	40.1
	Δ_α	0.5308	0.5308	0.5308	0.5308	0.5308	0.5308	0.5308

5.0 Comparative study

In this section, we define the relative efficiency function of the estimator, and analyze these functions to compare the relative performances of the estimators.

$$RE\left(\hat{\mu}_1(d):\tilde{\mu}_1\right)=\left[1-\frac{1}{2}\left(1-d^2\right)(1-\rho)+\frac{1}{2}(1-d)^2(1-\rho)\Delta^2\right]^{-1}. \quad (5.1)$$

We observed the following based on the expression in (5.1):

(i) if the non-sampling information is correct, i.e., $\Delta^2=0$, the

$$RE\left(\hat{\mu}_1(d):\tilde{\mu}_1\right)=\frac{2}{2-\left(1-d^2\right)(1-\rho)}>1$$

5.1 Comparing RE against UE

The relative efficiency of $\hat{\mu}_1(d)$ compared to $\tilde{\mu}_1$ is denoted by $RE(\hat{\mu}_1(d):\tilde{\mu}_1)$ and is obtained as

And $\hat{\mu}_1(d)$ is more efficient than $\tilde{\mu}_1$. Thus, under the null hypothesis the biased estimator, RE performs better than the unbiased estimator, UE.

(ii) if the non-sampling information is incorrect, i.e., $\Delta^2>0$ we study the expression in (5.1) as a function of Δ^2 for a fixed d-value. As a function of Δ^2, the expression in (4.1) is a decreasing function with its maximum value

$\frac{2}{2-\left(1-d^2\right)\left(1-\rho\right)}$ (>1) at Δ²=0 and minimum value 0 at Δ²=+∞.

5.2 comparing PTE against UE

$$E_3\left(\alpha;\Delta^2,\rho\right)=RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right)=\frac{MSE\left(\tilde{\mu}_1\right)}{MSE\left(\hat{\mu}_1^{PTE}\right)}=\left[1+g\left(\Delta^2\right)\right]^{-1}, \qquad (5.2)$$

Where

$$g\left(\Delta^2\right)=\frac{1}{2}(1-d)(1-\rho)\Delta^2\left[2H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)-(1+d)H_{5,v}\left(\frac{1}{5}F_{1,v}(\alpha),\Delta^2\right)\right]$$
$$-\frac{1}{2}\left(1-d^2\right)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right) \qquad (5.3)$$

For any fixed $d\,(0\le d\le 1)$ and at a fixed level of significance α. As $F_\alpha\to\infty$

$$RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right)\to\left[1-\frac{1}{2}\left(1-d^2\right)(1-\rho)+\frac{1}{2}(1-d)^2(1-\rho)\Delta^2\right]^{-1}. \qquad (5.4)$$

Which the relative efficiency is of $\hat{\mu}_1(d)$ compared to $\tilde{\mu}_1$. on the other hand, $F_\alpha\to 0,\ RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right)\to 1$, this means the relative efficiency of the PTE

$$\left[1-\frac{1}{2}\left(1-d^2\right)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\right]^{-1}\ge 1, \qquad (5.5)$$

Which is the maximum value of the relative efficiency. Thus the relative efficiency function monotonically decreases crossing the 1-line for Δ²-value between $\frac{1+d}{2}$

$$\Delta_*^2=\frac{(1+d)}{\left\{2-(1+d)\dfrac{H_{5,v}\left(\frac{1}{5}F_{1,v}(\alpha),\Delta^2\right)}{H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)}\right\}}. \qquad (5.6)$$

Where Δ_*^2 lies in the interval $\left(\frac{1+d}{2},\frac{1+d}{1-d}\right)$. This means that

$RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right)\gtreqless 1$ According as $\Delta^2\lesseqgtr\Delta_*^2$ $\Delta^2\to\infty,\ RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right)\to 1.$

Now, we consider the relative efficiency of the PTE compared to the UE .it is given by

is the same as the unrestricted estimator $\tilde{\mu}_1$. Note that under the null hypothesis,
Δ²=0, and the relative efficiency expression (4.2) equals

and $\frac{1+d}{1-d}$,0 a minimum for some $\Delta^2=\Delta^2_{\min}$ and then monotonically increases, to approach the unit value from below. The relative efficiency of the preliminary test estimator equals unity whenever

Finally, as $\Delta^2\to\infty,\ RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right)\to 1.$ Thus, the preliminary test estimator is more efficient than the unrestricted estimator whenever $\Delta^2<\Delta_*^2$, otherwise $\tilde{\mu}_1$ is more efficient. As for the relative efficiency of $\hat{\mu}_1^{PTE}(d)$ compared to $\hat{\mu}_1(d)$ we have

$$RE\left(\hat{\mu}_1^{PTE}:\hat{\mu}_1\right)=\left(1-\frac{1}{2}\left(1-d^2\right)(1-\rho)+\frac{1}{2}(1-d)^2(1-\rho)\Delta^2\right)\left[1+g\left(\Delta^2\right)\right]^{-1} \qquad (5.7)$$

Where

$$g\left(\Delta^2\right)=\frac{1}{2}(1-d)(1-\rho)\Delta^2\left[2H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right)-(1+d)H_{5,v}\left(\frac{1}{5}F_{1,v}(\alpha),\Delta^2\right)\right]$$
$$-\frac{1}{2}\left(1-d^2\right)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),\Delta^2\right) \qquad (5.8)$$

Under the null-hypothesis,

$$RE\left(\hat{\mu}_1^{PTE}:\hat{\mu}_1\right)=1-\frac{1}{2}\left(1-d^2\right)(1-\rho)\left[1-\frac{1}{2}\left(1-d^2\right)(1-\rho)H_{3,v}\left(\frac{1}{3}F_{1,v}(\alpha),0\right)\right]^{-1}$$

$$\ge 1-\frac{1}{2}\left(1-d^2\right)(1-\rho). \qquad (5.9)$$

At the same time we consider the result at (5.5), in combination, we obtain

$$1-\frac{1}{2}(1-d^2)(1-\rho)\le RE\left(\hat{\mu}_1^{PTE}(d):\hat{\mu}_1\right)\le 1\le RE\left(\hat{\mu}_1^{PTE}:\tilde{\mu}_1\right). \quad (5.10)$$

For general $\Delta^2>0,$ we have $RE\left(\hat{\mu}_1^{PTE}:\hat{\mu}_1\right)\stackrel{<}{\underset{>}{=}}1$ According as

$$\Delta^2 \stackrel{<}{\underset{>}{=}} \frac{\frac{1}{2}(1+d)\left[1-H_{3,\nu}\left(\frac{1}{3}F_{1,\nu}(\alpha),\Delta^2\right)\right]}{\frac{1}{2}(1-d)-2H_{3,\nu}\left(\frac{1}{3}F_{1,\nu}(\alpha),\Delta^2\right)-(1+d)H_{5,\nu}\left(\frac{1}{5}F_{1,\nu}(\alpha),\Delta^2\right)} \quad (5.11)$$

Finally, as $\Delta^2\to\infty,$ $RE\left(\hat{\mu}_1^{PTE}:\hat{\mu}(d)\right)\to 0.$ thus, except for a small interval around 0, $\hat{\mu}_1^{PTE}(d)$ is more efficient than $\hat{\mu}_1(d).$

$$E_4\left(\Delta^2;\rho\right)=RE\left(\hat{\mu}_1^{SE}:\tilde{\mu}_1\right)=\left[1+\frac{K^2}{\pi}(1-\rho)\left(1-2e^{-\frac{\Delta^2}{2}}\right)\right]^{-1}. \quad (5.12)$$

Under the null-hypothesis $\Delta^2=0$, we have

$$RE\left(\hat{\mu}_1^{SE}:\tilde{\mu}_1\right)=\left[1-\frac{K^2}{\pi}(1-\rho)\right]^{-1}\ge 1. \quad (5.13)$$

In general, $RE\left(\hat{\mu}_1^{SE}:\tilde{\mu}_1\right)$ decreases from $\left[1-\frac{K^2}{\pi}(1-\rho)\right]^{-1}$ at $\Delta^2=0$ and crosses the 1-line at $\Delta^2=\ln 4$ and then goes to the minimum value

$$\left[1+\frac{K^2}{\pi}(1-\rho)\right]^{-1} \text{ As } \Delta^2=\infty. \quad (5.14)$$

Thus, the loss of efficiency of $\hat{\mu}_1^{SE}$ relative to $\tilde{\mu}_1$ is

$$1-\left[1+\frac{K^2}{\pi}(1-\rho)\right]^{-1} \quad (5.15)$$

While the gain in efficiency is

$$\left[1-\frac{K^2}{\pi}(1-\rho)\right]^{-1} \quad (5.16)$$

Which is achieved at $\Delta^2=0$. Thus, for $\Delta^2<\ln 4,$ $\hat{\mu}_1^{SE}$ performs better than $\tilde{\mu}_1$,otherwise $\tilde{\mu}_1$ performs better. The property of $\hat{\mu}_1^{SE}$ is similar to the preliminary test estimator but does not depend on the level of significance.

Table (5.1) shows the maximum $\left(E^{\max}\right)$ relative efficiency of the SE at $\Delta^2=0$, the minimum efficiency SE $(E_{\min})$. the value of Δ at which the minimum efficiency SE occurs $(\Delta_{\min})$ and the intersection point between SE and UE (Δ_S) which equal to ln 4=1.3863. This means when $0\le\Delta^2\le 1.3863$ the SE dominates UE but outside the interval the UE dominates SE.

5.3 Comparing SE against UE

The relative efficiency of $\hat{\mu}_1^{SE}$ compared to $\tilde{\mu}_1$ is given by

5.4 Comparing SE against PTE

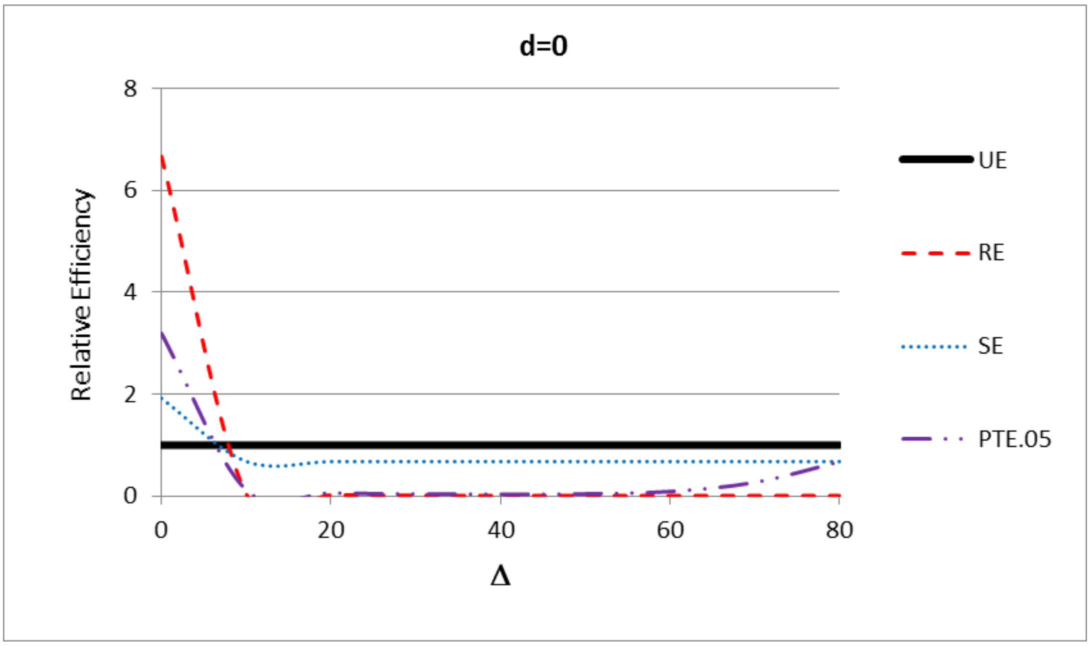
To compare the relative performances of the SE and the PTE, first note that the SE is superior to PTE when the null hypothesis is true and the level of significance, α is not too large. This is regardless of the value of the coefficient of distrust d.

However, as the value of Δ increases and or α grows larger the relative efficiency picture changes. Tables (5.2) provide the maximum relative efficiency at $\Delta^2=0$ for the RE (E1), PTE (E3), and SE(E4) relative to UE and the intersection relative efficiency $\left(E_{\Delta\alpha}\right)$ of the PTE and SE and the Δ_α -values at which the intersection occurs. For example, using Table (5.2) if n=5, ρ=0.3 and d=0 with α=0.05 and Δ∈[0, 0.5232], then PTE dominates SE, but outside the interval SE dominates PTE. If $\Delta^2=0$ the $\hat{\mu}_1(d)$ his large efficiency ($E_1=1.5385$) than $\hat{\mu}_1^{PTE}\left(E_3=1.3943\right)$ and $\hat{\mu}_1^{SE}\left(E_4=1.2453\right)$ and the efficiency at intersection point $\left(\Delta_{0.05}=0.5232\right)$ is equal to 1.1189. Also, note that for large values of α and the sample sizes n, SE dominates PTE uniformly, i.e., when Δ_α =0 in Table (5.2). Figures (5.1) display the behavior of the RE, PTE and SE for different values of α, d when $n=5$ and $\rho=-0.7$.

5.5 Conclusion

In this paper, we develop a general theory of shrinkage estimation in family of bivariate normal distribution with equal marginal variances from a sample of size *n*.

It is also showed that shrinkage estimator is asymptotically optimal in its own class and superior performance compared to the classical estimators. Its asymptotic optimality did not either depend on the specific distribution assumed (bivariate normal distribution) nor on the implicit assumption that with equal marginal variances from a sample of size n. The considered shrinkage estimators have higher relative efficiency than the classical estimators specially when the estimated value of the mean is closer to true one. In addition, when the null hypothesis is true and the level of significance, α is not too large. This is regardless of the value of the coefficient of distrust d. Naturally it is found that as coefficient of distrust arises the shrinkage estimators tend to become superior compared with their competitors.



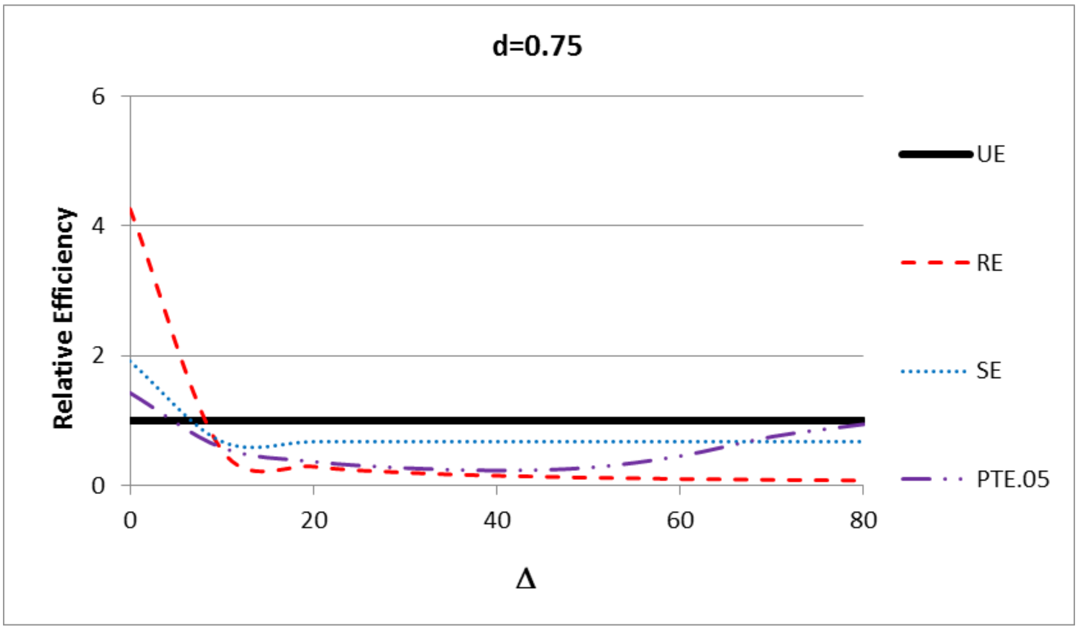


Figure 5-1: Graph of the Relative Efficiency of RE, PTE and SE for selected values of d and α for n=5 and ρ=-0.7

Table 5.1: Maximum and minimum efficiency of SE

n/ρ	R.E.	-0.9	-0.7	-0.1	0.1	0.3	0.7	0.9
5	$E^{\max}$	2.1489	1.9171	1.4483	1.3391	1.2453	1.0922	1.029
	$E_{\min}$	0.6517	0.6765	0.7637	0.798	0.8355	0.9222	0.9726
	$\Delta_{\min}$	17.1	17.1	17.1	17.1	17.1	17.1	17.1
	Δ_s	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863
15	$E^{\max}$	2.4032	2.0939	1.5107	1.3823	1.2741	1.1016	1.0317
	$E_{\min}$	0.6314	0.6569	0.7474	0.7834	0.823	0.9156	0.9702
	$\Delta_{\min}$	17.1	17.1	17.1	17.1	17.1	17.1	17.1
	Δ_s	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863
25	$E^{\max}$	2.4547	2.1288	1.5223	1.3903	1.2793	1.1032	1.0322
	$E_{\min}$	0.628	0.6536	0.7446	0.7809	0.8208	0.9145	0.9698
	$\Delta_{\min}$	17.1	17.1	17.1	17.1	17.1	17.1	17.1
	Δ_s	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863
35	$E^{\max}$	2.4769	2.1436	1.5272	1.3936	1.2815	1.1039	1.0324
	$E_{\min}$	0.6266	0.6522	0.7435	0.7798	0.8199	0.914	0.9696
	$\Delta_{\min}$	17.1	17.1	17.1	17.1	17.1	17.1	17.1
	Δ_s	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863
40	$E^{\max}$	2.4838	2.1483	1.5287	1.3946	1.2822	1.1041	1.0325
	$E_{\min}$	0.6261	0.6518	0.7431	0.7795	0.8197	0.9138	0.9695
	$\Delta_{\min}$	17.1	17.1	17.1	17.1	17.1	17.1	17.1
	Δ_s	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863
45	$E^{\max}$	2.4892	2.1519	1.5299	1.3955	1.2827	1.1043	1.0325
	$E_{\min}$	0.6258	0.6514	0.7428	0.7793	0.8195	0.9137	0.9695
	$\Delta_{\min}$	17.1	17.1	17.1	17.1	17.1	17.1	17.1
	Δ_s	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863	1.3863

Table 5.2: Maximum relative efficiencies of RE (E_1), PTE (E_3) and SE (E_4) and the intersecting efficiencies for PTE and SE ($E_{\Delta\alpha}$) for each α with corresponding Δ-values (E_α) for d=0 and n=5

α / n_2	R.E.	-0.9	-0.7	-0.1	0.1	0.3	0.5	0.9
0.05	E_1	20	6.6667	2.2222	1.8182	1.5385	1.3333	1.0526
	E_3	4.3021	3.1924	1.7998	1.5713	1.3943	1.2531	1.0421
	E_4	2.1489	1.9171	1.4483	1.3391	1.2453	1.1637	1.029
	$E_{\Delta_{0.05}}$	1.4055	1.348	1.2005	1.1583	1.1189	1.0822	1.0154
	$\Delta_{0.05}$	0.5232	0.5232	0.5232	0.5232	0.5232	0.5232	0.5232
0.1	E_1	20	6.6667	2.2222	1.8182	1.5385	1.3333	1.0526
	E_3	2.6828	2.2791	1.5702	1.4227	1.3006	1.1977	1.0341
	E_4	2.1489	1.9171	1.4483	1.3391	1.2453	1.1637	1.029
	$E_{\Delta_{0.1}}$	1.6522	1.5461	1.2963	1.23	1.1702	1.1159	1.0212
	$\Delta_{0.1}$	0.2804	0.2804	0.2804	0.2804	0.2804	0.2804	0.2804
0.25	E_1	20	6.6667	2.2222	1.8182	1.5385	1.3333	1.0526
	E_3	1.5046	1.4287	1.2409	1.1889	1.141	1.0968	1.018
	E_4	2.1489	1.9171	1.4483	1.3391	1.2453	1.1637	1.029
	$E_{\Delta_{0.25}}$	1.5046	1.4287	1.2409	1.1889	1.141	1.0968	1.018
	$\Delta_{0.25}$	0	0	0	0	0	0	0
0.35	E_1	20	6.6667	2.2222	1.8182	1.5385	1.3333	1.0526
	E_3	1.2669	1.2323	1.1389	1.1109	1.0841	1.0587	1.0112
	E_4	2.1489	1.9171	1.4483	1.3391	1.2453	1.1637	1.029
	$E_{\Delta_{0.35}}$	1.2669	1.2323	1.1389	1.1109	1.0841	1.0587	1.0112
	$\Delta_{0.35}$	0	0	0	0	0	0	0
0.5	E_1	20	6.6667	2.2222	1.8182	1.5385	1.3333	1.0526
	E_3	1.1017	1.09	1.0564	1.0457	1.0352	1.0249	1.0049
	E_4	2.1489	1.9171	1.4483	1.3391	1.2453	1.1637	1.029
	$E_{\Delta_{0.5}}$	1.1017	1.09	1.0564	1.0457	1.0352	1.0249	1.0049
	$\Delta_{0.5}$	0	0	0	0	0	0	0

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