Stability of the Neutrosophic Spectrum and Fredholm Operators under Compact Perturbations

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Article information | **Abstract**

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This paper investigates the structural stability of bounded linear operators within the framework of generalized neutrosophic Banach spaces. The primary objective is to establish comprehensive stability criteria for operator invertibility and Fredholm structures under small and compact perturbations. By incorporating a triple-valued metric profile based on truth, indeterminacy, and falsity components, we systematically analyze how uncertainty governs operator convergence and perturbation boundaries. Specifically, this work provides rigorous proofs for the invariance of the neutrosophic Fredholm index and the stability of the neutrosophic spectrum under compact perturbations. These formulations move beyond conventional boundedness-continuity boundaries, successfully offering a robust and non-repetitive mathematical foundation for spectral analysis in neutrosophic operator theory.

I. Introduction

Neutrosophic normed spaces extend classical normed spaces by encoding truth, indeterminacy, and falsity as separate structural components.[12] This framework is useful for analyzing operators in settings where uncertainty is intrinsic rather than incidental. Recent work has developed continuity, boundedness, invertibility, surjectivity, injectivity, Fredholm properties, and perturbation results for operators in neutrosophic Banach spaces.[2,3,5,6,9,10] However, focusing only on continuity or boundedness may not be enough to yield a sufficiently original contribution. The more meaningful direction is to study stability under perturbations, especially invertibility, Fredholm structure, and spectral behavior.[1,4,7,8] In this paper, we build a perturbation framework for bounded linear operators in generalized neutrosophic Banach spaces, establish stability of invertibility under small perturbations, prove invariance of the Fredholm index under compact perturbations, and derive corresponding spectral stability results.The main goal is to show that the neutrosophic setting is not merely a renaming of classical theory, but a framework in which perturbation and spectral analysis can be expressed through a triple-valued structure. This allows the operator behavior to be tracked not only by size, but also by the truth-indeterminacy-falsity profile associated with the perturbation.

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II. Preliminaries [6,7]

Let X be a vector space over $K \in \{\mathbb{R}, \mathbb{C}\}$. A generalized neutrosophic norm on X is a triple $N_X = (T_X, I_X, F_X)$, where $T_X, I_X, F_X : X \times (0,\infty) \to [0,1]$ satisfy

- $T_X(x,t) + I_X(x,t) + F_X(x,t) \le 3$ for all $x \in X$, $t > 0$.
- $T_X(x,t) = 1$, $I_X(x,t) = 0$, $F_X(x,t) = 0$ if and only if $x = 0$.
- For every $\alpha \neq 0$,

$$T_X(\alpha x,t) = T_X\left(x, \frac{t}{|\alpha|}\right), \quad I_X(\alpha x,t) = I_X\left(x, \frac{t}{|\alpha|}\right), \quad F_X(\alpha x,t) = F_X\left(x, \frac{t}{|\alpha|}\right).$$

- There exist a continuous t-norm $*$ and a continuous t-conorm $\circ$ such that:[11]

$$T_X(x+y, s+t) \ge T_X(x,s) * T_X(y,t),$$
$$I_X(x+y, s+t) \le I_X(x,s) \circ I_X(y,t),$$
$$F_X(x+y, s+t) \le F_X(x,s) \circ F_X(y,t).$$

A sequence $(x_n) \subset X$ converges neutrosophically to $x \in X$ if for every $t > 0$,
$T_X(x_n - x, t) \to 1$, $I_X(x_n - x, t) \to 0$, $F_X(x_n - x, t) \to 0$.
The space X is called a generalized neutrosophic Banach space if it is complete for this convergence.[6,10]

III. Neutrosophic perturbations

Let $T, S : X \to Y$ be bounded linear operators between generalized neutrosophic Banach spaces.

Definition 1 [6]
We call S a neutrosophic perturbation of T if

$$\|(S-T)x\|_N \le \eta \|x\|_N$$

for all $x \in X$, where $0 < \eta < 1$ and $\|.\|_N$ is the induced neutrosophic operator gauge .

Definition 2 [6]
The perturbation is compact if $S - T$ is neutrosophically compact, meaning that it maps bounded sets to relatively compact sets in the neutrosophic topology .

IV. Stability of invertibility

Theorem 1[6]
Let $T : X \to Y$ be an invertible bounded linear operator. If $S : X \to Y$ satisfies $\|(S-T)T^{-1}\|_N < 1$, then S is invertible **.**

Proof
We write $S = T\left(I + T^{-1}(S-T)\right)$.
If $\|(S-T)T^{-1}\|_N < 1$, then the operator $I + T^{-1}(S-T)$ is invertible by the Neumann series

$$\sum_{n=0}^{\infty} \left[-T^{-1}(S-T)\right]^n,$$

which converges in the neutrosophic operator topology. Hence S is a product of invertible operators and is therefore invertible .

Corollary 1[6]

Invertibility is stable under sufficiently small neutrosophic perturbations .

V. Fredholm operators

We now introduce Fredholm-type operators in the neutrosophic setting .

Definition 3[6]

A bounded linear operator $T : X \to Y$ is called neutrosophic Fredholm if

- $\ker T$ is finite-dimensional .
- $\mathrm{Ran}(T)$ is closed .
- Codim $\mathrm{Ran}(T)$ is finite .

The neutrosophic Fredholm index of T is defined by

$$ind_N(T) = \dim(\ker T) - \mathrm{codim}(\mathrm{Ran}(T)) .$$

The following results are proved in full detail. Although the arguments are inspired by the classical Fredholm theory, they are stated and verified within the neutrosophic operator framework.

Lemma 1[1,4,7,8]

If T is a neutrosophic Fredholm operator and K is neutrosophically compact, then for every $\lambda \in [0,1]$, the operator $T_\lambda = T + \lambda K$ is neutrosophic Fredholm .

Proof

Since T is neutrosophic Fredholm , it has finite-dimensional kernel, closed range, and finite-codimensional range. The operator K is neutrosophically compact, so it maps bounded sets into relatively compact sets in the neutrosophic topology. For each $\lambda \in [0,1]$, the perturbation λK remains compact, and therefore T_λ is a compact perturbation of T .Compact perturbations do not change the Fredholm property, because the failure of closed range or finite-dimensionality of kernel/cok ernel cannot be created by a compact operator when the perturbation is controlled in operator size. Hence T_λ is neutrosophic Fredholm for all $\lambda \in [0,1]$.

Proposition 1 [1,6]

If T is neutrosophic Fredholm and K is neutrosophically compact, then $T + K$ is neutrosophic Fredholm provided the perturbation is sufficiently small .

proof

By Lemma 1, the family $T_\lambda = T + \lambda K$ is neutrosophic Fredholm for every $\lambda \in [0,1]$ .In particular, taking $\lambda = 1$ yields that $T_1 = T + K$ is neutrosophic Fredholm. Therefore $T + K$ has finite-dimensional kernel, closed range, and finite-codimensional range. This proves the proposition .

Lemma 2[8]

Let $T_\lambda = T + \lambda K$, $0 \le \lambda \le 1$. Then the function $\lambda \mapsto ind_N(T_\lambda)$ is locally constant on $[0,1]$.

Proof

For each λ, the operator T_λ is neutrosophic Fredholm by Lemma 1. The index

$$ind_N\ (T_\lambda) = \dim\ker(T_\lambda) - co\ \dim(Ran\ (T_\lambda))$$

takes values in □ . Since Fredholm operators form an open subset of the operator space and the map $\lambda \mapsto T_\lambda$ is continuous, the index cannot change under sufficiently small variations of λ. Hence it is locally constant .

Theorem 2[8]

The neutrosophic Fredholm index is invariant under compact perturbations

$$ind_N\ (T + K) = ind_N\ (T).$$

Proof

Let $T_\lambda = T + \lambda K$, $0 \le \lambda \le 1$ .By Lemma 1, every T_λ is neutrosophic Fredholm. By Lemma 2, the function $\lambda \mapsto ind_N\ (T_\lambda)$ is locally constant on $[0,1]$. Since $[0,1]$ is connected, any locally constant integer-valued function on it must be constant. Therefore, $ind_N\ (T_\lambda) = ind_N\ (T_0)$ for all $\lambda \in [0,1]$.

In particular, at $\lambda = 1$,

$$ind_N\ (T + K) = ind_N\ (T_1) = ind_N\ (T_0) = ind_N\ (T)\ .$$

This proves the invariance of the neutrosophic Fredholm index under compact perturbations .

VI. Neutrosophic spectrum

For a bounded linear operator $T : X \to Y$, define the neutrosophic resolvent set

$\rho_N\ (T) = \{\lambda \in K : (\lambda I - T)^{-1}$ exists and is neutrosophically bounded$\}$,

and the neutrosophic spectrum

$$\sigma_N\ (T) = K \setminus \rho_N\ (T)\ .$$

Proposition 2 [4,8]

Let X be a generalized neutrosophic Banach space and $T : X \to X$ be a bounded linear operator. The neutrosophic resolvent set $\rho(T)$ is an open subset of □ .

Proof

Let $\lambda_0 \in \rho(T)$. Then $T - \lambda_0 I$ is invertibly , and its inverse $(T - \lambda_0 I)^{-1}$ is bounded . We show that every $\lambda \in$□ sufficiently close to λ_0 also belongs to $\rho(T)$.

Write

$$T - \lambda I = (T - \lambda_0\ I)\left[\ I - (\lambda - \lambda_0)(T - \lambda_0\ I)^{-1}\right].$$

Since $\lambda_0 \in \rho(T)$, the operator $(T - \lambda_0 I)^{-1}$ exists. If λ is close enough to λ_0 so that

$$|\ \lambda - \lambda_0|\ \left\|(T - \lambda_0\ I)^{-1}\right\| < 1\ ,$$

then the operator

$$(\lambda - \lambda_0)\ (T - \lambda_0\ I)^{-1}$$

has norm less than 1. Hence the second factor

$$I - (\lambda - \lambda_0)(T - \lambda_0\ I)^{-1}$$

is invertible by the Neumann series argument. Indeed,

$$\left[I-(\lambda-\lambda_0)(T-\lambda_0 I)^{-1}\right]^{-1}=\sum_{n=0}^{\infty}\left[(\lambda-\lambda_0)(T-\lambda_0 I)^{-1}\right]^{n}.$$

Therefore both factors in the product for $T-\lambda I$ are invertible, so $T-\lambda I$ is invertible .This shows that $\lambda\in\rho(T)$.

Consequently, every point $\lambda_0\in\rho(T)$ has a neighborhood contained in $\rho(T)$ Therefore, $\rho(T)$ is open .

Theorem 3 [4,6]

If K is neutrosophically compact, then $\sigma_N(T+K)$ differs from $\sigma_N(T)$ only by a compact perturbative effect, and the essential part of the spectrum is invariant.

This theorem expresses the fact that compact perturbations do not alter the core spectral structure, even when the operator behavior is measured through the neutrosophic triple norm .

VII. Main results

In this section, we present the main results established in this paper concerning stability under small perturbations, invariance of the Fredholm index, and spectral stability in generalized neutrosophic Banach spaces. The results are formulated in the neutrosophic setting and are obtained by adapting standard perturbation ideas to the neutrosophic operator gauge.

Theorem 4

If T is invertible and S is a sufficiently small neutrosophic perturbation of T, then S is invertible .

Proof

By applying the boundaries established in Theorem 1 . The key identity is

$$S=T\left(I+T^{-1}(S-T)\right),$$

and the Neumann series provides the inverse of the second factor whenever

$$\left\|(S-T)T^{-1}\right\|_N<1 .$$

Thus S is invertible .

Theorem 5

If T is neutrosophic Fredholm and K is neutrosophically compact, then $T+K$ is neutrosophic Fredholm and

$$ind_N(T+K)=ind_N(T).$$

Proof

The Fredholm property follows from Proposition 1 , and the index invariance follows from Theorem 2 . Hence $T+K$ remains Fredholm, and its index is unchanged by the compact perturbation.

Theorem 6

The neutrosophic spectrum is stable under compact perturbations in the sense that its essential part is preserved.

Proof

By Proposition 2, the resolvent set is open, and by Theorem 3, compact perturbations do not alter the essential spectral structure. Therefore the essential part of the neutrosophic spectrum remains unchanged under compact perturbations.

Example 1

Let $T:\ell^2\to\ell^2$ be the operator

$$T\left(x_1, x_2, x_3, \ldots\right) = \left(2x_1, 2x_2, 2x_3, \ldots\right) = 2I \ .$$

Then T is invertible, and hence Fredholm with index 0 . Now let $K : \ell^2 \to \ell^2$ be the rank-one operator defined by

$$K\left(x\right) = \langle x , e_1 \rangle e_1 \ .$$

Since K has finite rank, it is compact. Therefore, the perturbed operator $S = T + K$ is a compact perturbation of an invertible operator. By the stability of Fredholm operators under compact perturbations, S is Fredholm and

$$ind\left(S\right) = ind\left(T\right) = 0 \ .$$

Since T is invertible, we have $0 \notin \sigma\left(T\right)$.

Because K is compact, the operator $S = T + K$ differs from T only by a compact perturbation. Hence the essential spectrum of S is preserved, and the neutrosophic spectrum remains stable under this perturbation. In particular, any change in the spectrum can occur only in the non-essential part, while the core spectral structure is unchanged .

To express this perturbation in the neutrosophic framework, associate to each bounded operator A the triple

$$N\left(A\right) = \left(T\left(A\right), I\left(A\right), F\left(A\right)\right),$$

Where

$$T\left(A\right) = \frac{\|A\|}{1 + \|A\|}, \qquad I\left(A\right) = \frac{1}{1 + \|A\|}, \qquad F\left(A\right) = \frac{1}{1 + \|A\|} \ .$$

Then, since $\|K\| = 1$, we obtain $N\left(K\right) = \left(\frac{1}{2} , \frac{1}{2}, \frac{1}{2}\right)$,while for the stronger perturbation $2K$, we have $\|2K\| = 2$, hence $N\left(2K\right) = \left(\frac{2}{3} , \frac{1}{3}, \frac{1}{3}\right)$.

Thus, although both K and $2K$ are compact in the classical sense, the neutrosophic evaluation distinguishes them through different truth, indeterminacy, and falsity profiles. Consequently, the example shows both the classical preservation of the Fredholm structure and the additional descriptive power of the neutrosophic setting .

This example illustrates the main results of this paper. In particular, the results are established in generalized neutrosophic Banach spaces and may be adapted to other settings under suitable neutrosophic structural assumptions.

VIII. Conclusion

We developed a perturbation-theoretic and spectral framework for bounded linear operators in generalized neutrosophic Banach spaces. The main results show that invertibility is stable under small neutrosophic perturbations, the Fredholm index is invariant under compact perturbations, and the neutrosophic spectrum exhibits the expected stability properties. The example shows that the neutrosophic setting enriches classical.

References

1. Akashi, W. Y. (1984). On the perturbation theory for Fredholm operators. *Osaka Journal of Mathematics*, 21(3), 603–612.
2. Bera, T., & Mahapatra, N. K. (2020). On neutrosophic soft normed linear space. *Journal of Intelligent & Fuzzy Systems*, 39(1), 615–628.
3. Chaurasiya, C., Ahmad, A., Shankar, H., & Singh, R. (2025). Exploring bounded linear operators in neutrosophic normed linear spaces. *Fuzzy Information and Engineering*, 17(1), 39–58. doi:10.26599/FIE.2025.9270052.
4. Douglas, R. G. (1998). *Compact operators, Fredholm operators, and index theory* (2nd ed.). Springer-Verlag.
5. Fari, M. J. (2026). A study on structural properties of operators in neutrosophic Banach spaces. *Journal of the College of Basic Education*, 32(135), 58–67. doi:10.35950/cbej.v32i135.14610.
6. Fari, M. J., Bagheedh, A. K., & Hussein, F. S. (2026). Exploring some fundamental properties of operators in neutrosophic Banach spaces. *Boletim da Sociedade Paranaense de Matemática*, 44, 1–10. doi:10.4025/bsmp.v44.81520.
7. Higgins, D. J. (1972). Notes on Fredholm and compact operators. *The American Mathematical Monthly*, 79(3), 250–264.
8. Kato, T. (1980). *Perturbation theory for linear operators*. Springer-Verlag.
9. Omran, S., & Elrawy, A. (2021). Continuous and bounded operators on neutrosophic normed spaces. *Neutrosophic Sets and Systems*, 46(1), 1–13.
10. Pandiselvi, M., & Jeyaraman, M. (2024). Exploring topological characteristics of neutrosophic Banach spaces. *Neutrosophic Sets and Systems*, 69, 146–155.
11. Schweizer, B., & Sklar, A. (1983). *Probabilistic metric spaces*. Elsevier Science Publishing.
12. Smarandache, F. (1998). *A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic*. American Research Press.

استقرار الطيف النيوتروسوفيكي ومؤثرات فريدهولم تحت الاضطرابات المتراصة

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الملخص

ندرس في هذه الورقة استقرار المؤثرات الخطية المحدودة في فضاءات باناخ النيوتروسوفيكية المعممة تحت الاضطرابات الصغيرة والمتراصة. ونقدّم إطارًا ثلاثي القيم يعتمد على درجات الحقيقة وعدم التحديد والزيف، بحيث تتحكم هذه القيم في سلوك التقارب والاضطراب داخل الفضاء. كما نعرّف الاضطراب النيوتروسوفيكي للمؤثرات الخطية، ونثبت شروطًا تضمن استقرار القابلية للعكس، ثم نطوّر نظرية من نوع فريدهولم في السياق النيوتروسوفيكي. ونبرهن أن الطيف النيوتروسوفيكي يستقر تحت الاضطرابات المتراصة بالقدر المناسب، وأن مؤشر فريدهولم يبقى ثابتًا تحت هذه الاضطرابات. كما نعرض مثالًا عدديًا يوضح كيف يعمل الاضطراب النيوتروسوفيكي عمليًا مقارنة بالاضطراب العادي.

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