On The Completion of Hausdorff Topological Vector Spaces Using Nets

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ABSTRACT

This paper contains a direct proof of the result asserting that every Hausdorff topological vector space has a completion. The proof here is different from known proofs of this result, since it uses only the convergence of nets and their basic properties. The completion is constructed as equivalence classes of Cauchy sequences.

KEYWORDS

Topological vector spaces, Completion, Nets, Subnets, Cauchy nets, Directed sets, Convergence.

1. INTRODUCTION

A topological vector space (TVS) is a real or complex vector space with topological structure makes the two vector space operations continuous, provided that the topology on the real (or complex) field is the usual topology. The theory of TVSs is a very important branch of mathematics. Soon after TVSs were introduced back in the 1930's, it was fount that they have a very rich theory and they are now considered as the corner stone in functional analysis.

Considered as a generalization of normed spaces, TVSs were expacted to have the completion property as in the case in normed spaces. In the literature, there are several proofs of this property. Some of those proofs are non-elementary and using advanced and involving tool of mathematical analysis as in [1] (Sec 3.3). A similar situation in [2], a proof is provided for the completion of commutative topological groups (Theorem 3.2.7) and the case for topological vector spaces follows immediately. Some others were only for special cases, see [3], Chap VI, Theorem 1 whire completion for locally convex topological vector spaces.

We thought of a natural proof generalizing the Cauchy sequence approach to normed spaces and metric spaces in general. Our proof is elementary and uses only the definition of TVSs and the convergence in them. To our knowledge, this approach was not persued in published literature concerning the theory of TVSs.

We have provided a quick review of convergence in TVSs in the second section. The proof of the main theorem is given in the third section.

2. NETS IN TOPOLOGICAL VECTOR SPACES

A *directed set* is a set I with a relation $\leq$ satisfying the following conditions:

- $(\forall \lambda \in I)\ \ \lambda \leq \lambda$;
- $\lambda_1 \leq \lambda_2,\ \lambda_2 \leq \lambda_3 \ \Rightarrow\ \lambda_1 \leq \lambda_3$;
- $(\forall \lambda_1, \lambda_2 \in I)\ \ (\exists \lambda \in I)\ \ \lambda_1 \leq \lambda, \lambda_2 \leq \lambda$.

A *net* in a set X is a mapping $P: I \to X$, where I is a directed set. A net is usually written as $(x_\lambda)_{\lambda\in I}$ where $x_\lambda = P(\lambda)$.

A *subnet* of $(x_\lambda)_{\lambda\in I}$ is a net $(x_{S(\mu)})_{\mu\in J}$, where J is a directed set and S is a mapping $S: J \to I$ satisfying the following conditions:

- $\mu_1, \mu_2 \in J,\ \mu_1 \le \mu_2 \ \Rightarrow\ S(\mu_1) \le S(\mu_2)$;
- $(\forall \lambda \in I)\ (\exists \mu \in J)\ \lambda \le S(\mu)$.

If X is a topological space, we say the net $(x_\lambda)_{\lambda\in I} \subseteq X$ *converges* to $x \in X$ if for any neighborhood N there is $\lambda_0 \in I$ such that $x_\lambda \in N$ whenever $\lambda_0 \le \lambda$; we denote this as $x_\lambda \to x$ or $\lim_{\lambda\in I} x_\lambda = x$ (usually we write $\lim_\lambda x_\lambda$ or $\lim x_\lambda$ if there is no ambiguity). We say that x is an *accumulation point* of the net $(x_\lambda)_{\lambda\in I}$ if for any neighborhood N and any $\lambda \in I$ there is $\lambda' \in I$ such that $\lambda \le \lambda'$ and $x_{\lambda'} \in N$. The point x is an accumulation point of the net $(x_\lambda)_{\lambda\in I}$ if and only if there is a subnet $(x_{S(\mu)})_{\mu\in J}$ of $(x_\lambda)_{\lambda\in I}$ such that $\lim_{\mu\in J} x_{S(\mu)} = x$.

In a topological space X, a point x is belongs to the closure of the set $A \subseteq X$ if and only if there is a net $(x_\lambda)_{\lambda\in I} \subseteq A$ such that $x_\lambda \to x$. If Y is a topological space and $f: X \to Y$ is a mapping, then f is continuous at $x_0 \in X$ if and only if $\lim_\lambda f(x_\lambda) = f(x_0)$ for any net $(x_\lambda)_{\lambda\in I}$ in X such that $\lim_\lambda x_\lambda = x_0$. A topological space is a Hausdorff space if and only if every net in this space converges at most to one point.

Let I, J be directed sets and define the following relation on the Cartesian product $I \times J$:

$$(\lambda_1, \mu_1) \le (\lambda_2, \mu_2) \ \text{ whenever } \ \lambda_1 \le \lambda_2 \ \textit{and} \ \mu_1 \le \mu_2.$$

The set $I \times J$ with this relation is a directed set.

Let X be a TVS (Topological Vector Space) and let $(x_\lambda)_{\lambda\in I}, (y_\mu)_{\mu\in J}$ be nets in X. We define the sum and the difference of these nets as

$$(x_\lambda + y_\mu)_{(\lambda,\mu)\in I\times J}, \qquad (x_\lambda - y_\mu)_{(\lambda,\mu)\in I\times J},$$

respectively. Moreover, if $(\alpha_\nu)_{\nu\in K}$ is a net of scalars, then the product of the nets $(\alpha_\nu)_{\nu\in K}$ and $(x_\lambda)_{\lambda\in I}$ is defined as:

$$(\alpha_\nu x_\lambda)_{(\lambda,\nu)\in I\times K}.$$

Now we have the following result:

Proposition 2.1 *Let $(x_\lambda)_{\lambda\in I}, (y_\mu)_{\mu\in J}$ be convergent nets in the TVS X and $(\alpha_\nu)_{\nu\in K}$ is a convergent net of scalars.*

1. The net $(x_\lambda + y_\mu)_{(\lambda,\mu)\in I\times J}$ is convergent and
$$\lim_{(\lambda,\mu)} (x_\lambda + y_\mu) = \lim_\lambda x_\lambda + \lim_\mu y_\mu.$$
2. The net $(x_\lambda - y_\mu)_{(\lambda,\mu)\in I\times J}$ is convergent and
$$\lim_{(\lambda,\mu)} (x_\lambda - y_\mu) = \lim_\lambda x_\lambda - \lim_\mu y_\mu.$$
3. The net $(\alpha_\nu x_\lambda)_{(\lambda,\nu)\in I\times K}$ is convergent and
$$\lim_{(\lambda,\nu)} \alpha_\nu x_\lambda = \left(\lim_\nu \alpha_\nu\right)\left(\lim_\lambda x_\lambda\right).$$

Proof. We prove (1) only, (2) and (3) are done similarly. Suppose that $x=\lim_\lambda x_\lambda,\ y=\lim_\mu y_\mu$ and let N be a neighborhood of $x+y$. There is a neighborhood N_0 of zero in X such that $x+y+N_0\subseteq N$ and there is a neighborhood M_0 of zero such that $M_0+M_0\subseteq N_0$.

Note that $x+M_0,\ y+M_0$ are neighborhoods of x,y, respectively. By convergence, there are $\lambda_0\in I,\mu_0\in J$ such that $x_\lambda\in x+M_0,\ y_\mu\in y+M_0$ whenever $\lambda_0\le\lambda,\mu_0\le\mu$. Thus,

$$x_\lambda+y_\mu\in x+y+M_0+M_0\subseteq x+y+N_0\subseteq N.$$

Hence, if $(\lambda_0,\mu_0)\le(\lambda,\mu)$, then $x_\lambda+y_\mu\in N$. Since N is arbitrary neighborhood, it follows that $(x_\lambda+y_\mu)_{(\lambda,\mu)\in I\times J}$ is convergent and $\lim_{(\lambda,\mu)}(x_\lambda+y_\mu)=x+y$. ■

Definition 2.2 *Let X be topological vector space. We say that the net $(x_\lambda)_{\lambda\in I}\subseteq X$ is a Cauchy net in X if for any neighborhood N of zero there is $\lambda_0\in I$ such that $x_{\lambda_1}-x_{\lambda_2}\in N$, whenever $\lambda_0\le\lambda_1,\ \lambda_0\le\lambda_2$.*

Remark 2.3

1. The net $(x_\lambda)_{\lambda\in I}$ in the TVS X is a Cauchy net if and only if $\lim_{(\lambda,\mu)\in I\times I}(x_\lambda-x_\mu)=0$.
2. If the net $(x_\lambda)_{\lambda\in I}$ is convergent, then it is a Cauchy net.

Proposition 2.4 *Let $(x_{S(\mu)})_{\mu\in J}$ be a subnet of the Cauchy net $(x_\lambda)_{\lambda\in I}$ in the TVS X. If $(x_{S(\mu)})_{\mu\in J}$ is convergent, then $(x_\lambda)_{\lambda\in I}$ is a convergent and* $\lim_{\lambda\in I}x_\lambda=\lim_{\mu\in J}x_{S(\mu)}$.

Proof. Let $x=\lim_{\mu\in J}x_{S(\mu)}$ and let N be a neighborhood of x. There is a neighborhood N_0 of 0 such that $x+N_0\subseteq N$ and there is a neighborhood M_0 of 0 such that $M_0+M_0\subseteq N_0$.
Since $(x_\lambda)_{\lambda\in I}$ is a Cauchy net, there is $\lambda_0\in I$ such that $x_{\lambda_1}-x_{\lambda_2}\in M_0$ if $\lambda_0\le\lambda_1,\lambda_0\le\lambda_2$. By convergence, there is $\mu_0\in J$ such that $x_{S(\mu)}\in x+M_0$, whenever $\mu_0\le\mu$. We can choose $\mu_1\in J$ such that $\mu_0\le\mu_1$ and $\lambda_0\le S(\mu_1)$. Thus, if $\lambda_0\le\lambda$, then

$$x_\lambda=x_{S(\mu_1)}+x_\lambda-x_{S(\mu_1)}\in x+M_0+M_0\subseteq x+N_0.$$

Hence, $x_\lambda\in N$, whenever $\lambda_0\le\lambda$. Thus, we have proved that $(x_\lambda)_{\lambda\in I}$ is convergent and $\lim_{\lambda\in I}x_\lambda=x$. ■

Corollary 2.5 *If a Cauchy net has a accumulation point, then it convergent and it converge to its accumulation point.*

3. Completion of Hausdorff Topological Vector Spaces

In this section, X will denote a Hausdorff TVS. Let X^{C} be the set of all Cauchy nets in X and let $\cong$ be the relation on X^{C} defined as follows

$$(x_\lambda)_{\lambda\in I}\cong(y_\mu)_{\mu\in J}\ \text{if the net}\ (x_\lambda-y_\mu)_{(\lambda,\mu)\in I\times J}\ \text{converges to zero.}$$

This is an equivalence relation on X^{C}. It is obvious from definition that $\cong$ is reflexive and symmetric. To show that it is transitive we suppose that

$$(x_\lambda)_{\lambda\in I}\cong(y_\mu)_{\mu\in J}\ \text{and}\ (y_\mu)_{\mu\in J}\cong(z_\nu)_{\nu\in K}.$$

Let N be a zero neighborhood and choose a zero neighborhoods M such that $M+M\subseteq N$. Since $\lim_{(\lambda,\mu)}(x_\lambda-y_\mu)=\lim_{(\mu,\nu)}(y_\mu-z_\nu)=0$, there are $(\lambda_1,\mu_1)\in I\times J$ and $(\mu'_1,\nu_1)\in J\times K$ such that $x_\lambda-y_\mu\in M$, whenever $(\lambda_1,\mu_1)\le(\lambda,\mu)$ and $y_\mu-z_\nu\in M$, whenever $(\mu'_1,\nu_1)\le(\mu,\nu)$. Hence, it follows easily that $x_\lambda-z_\nu\in N$, whenever $(\lambda_1,\nu_1)\le(\lambda,\nu)$ and hence $\lim_{(\lambda,\nu)}(x_\lambda - z_\nu) = 0$. Therefore, $(x_\lambda)_{\lambda\in I} \cong (z_\nu)_{\nu\in K}$.

We define the vector space $\tilde{X}$ as the quotient set $X^{\mathrm{C}}/\cong$ privided with the following two vector space operations:

$$[(x_\lambda)_{\lambda\in I}] + \left[(y_\mu)_{\mu\in J}\right] = \left[(x_\lambda + y_\mu)_{(\lambda,\mu)\in I\times J}\right], \quad c \cdot [(x_\lambda)_{\lambda\in I}] = [(c \cdot x_\lambda)_{\lambda\in I}].$$

The notation $[.]$ stands for $\cong$-equivalence class. These operations are well-defined and we will prove this only for addition, scalar multiplication can be done similarly and far more easily. If $[(x_\lambda)_{\lambda\in I}] = [(x'_{\lambda'})_{\lambda'\in I'}]$, $[(y_\mu)_{\mu\in J}] = [(y'_{\mu'})_{\mu'\in J'}]$, then

$$\lim_{(\lambda,\lambda')}(x_\lambda - x'_{\lambda'}) = \lim_{(\mu,\mu')}(y_\mu - y'_{\mu'}) = 0$$

Therefore, we have

$$\lim\left((x_\lambda + y_\mu) - (x'_{\lambda'} + y'_{\mu'})\right) = \lim\left((x_\lambda - x'_{\lambda'}) + (y_\mu - y'_{\mu'})\right) = 0.$$

This implies $\left[(x_\lambda + y_\mu)_{(\lambda,\mu)\in I\times J}\right] = \left[(x_{\lambda'} + y_{\mu'})_{(\lambda',\mu')\in I'\times J'}\right]$.

We can define a mapping $X \to \tilde{X},\ x \mapsto \tilde{x}$, where $\tilde{x} = [(x_\lambda)_{\lambda\in I}]$, the net $(x_\lambda)_{\lambda\in I}$ is a constant net defined by $x_\lambda = x\ (\forall\lambda \in I)$ for a fixed directed set I. Since X is Hausdorff space, this mapping is injective, and hence, X can be imbedded in $\tilde{X}$ as vector space by identifying every $x \in X$ with its corresponding equivalence class $\tilde{x}$ in $\tilde{X}$. Therefore, from now on, we will always consider $X \subseteq \tilde{X}$.

Now we need to define a vector topology on $\tilde{X}$. Let $\mathcal{N}$ be neighborhood base at in the TVS X consisting of balanced neighborhoods. For any $N \in \mathcal{N}$ we define

$$\widetilde{N} = \{\tilde{x} \in \tilde{X} : \exists (x_\lambda)_{\lambda\in I} \in \tilde{x} \text{ and } \exists \lambda_0 \in I \text{ such that } \lambda_0 \le \lambda \Rightarrow x_\lambda \in N\}.$$

Note that $N \subseteq \widetilde{N}$. Let $\widetilde{\mathcal{N}}$ be the family of all $\widetilde{N} \subseteq \tilde{X}$, where $N \in \mathcal{N}$; that is $\widetilde{\mathcal{N}} = \{\widetilde{N} : N \in \mathcal{N}\}$.

Theorem 3.1 *There is a unique vector topology on $\tilde{X}$ such that the family $\widetilde{\mathcal{N}}$ is a neighborhood base at zero. Moreover, under this topology, $\tilde{X}$ is a Hausdorff space and the TVS X is a dense subspace of $\tilde{X}$.*

Proof. Let $\widetilde{N} \in \widetilde{\mathcal{N}}$ and let $|\alpha| \le 1$. If $\tilde{x} \in \widetilde{N}$, then there is $(x_\lambda)_{\lambda\in I} \in \tilde{x}$ and $\lambda_0 \in I$ such that $x_\lambda \in N$, whenever $\lambda_0 \le \lambda$. Since N is balanced, $\alpha x_\lambda \in N$ for $\lambda_0 \le \lambda$. Hence, $\alpha\tilde{x} = [(\alpha x_\lambda)_{\lambda\in I}] \in \widetilde{N}$ and we conclude that $\widetilde{N}$ is balanced since α is arbitrary.

Let $\tilde{x} \in \tilde{X}$ and let $N \in \mathcal{N}$. There is $N_1 \in \mathcal{N}$ such that $N_1 + N_1 \subseteq N$. Since $\tilde{x} = [(x_\lambda)_{\lambda\in I}]$, there is $\lambda_0 \in I$ such that $x_\lambda - x_{\lambda_0} \in N_1$, whenever $\lambda_0 \le \lambda$, since $(x_\lambda)_{\lambda\in I}$ is a Cauchy net. Since N_0 is absorbent, there is $r_0 > 0$ such that $\alpha x_{\lambda_0} \in N_0$, whenever $|\alpha| \le 1$. If we choose $r = \min\{r_0, 1\}$, then, since N_1 is balanced, we have

$$|\alpha| \le r \textit{ and } \lambda_0 \le \lambda \text{ imply } \alpha x_\lambda = \alpha(x_\lambda - x_{\lambda_0}) + \alpha x_{\lambda_0} \in N_1 + N_1 \subseteq N.$$

Therefore, $\alpha\tilde{x} \in \widetilde{N}$, whenever $|\alpha| \le r$ and this means $\widetilde{N}$ is absorbent.

If $N \in \mathcal{N}$, the there is $N_1 \in \mathcal{N}$ such that $N_1 + N_1 \subseteq N$ and this implies $\widetilde{N}_1 + \widetilde{N}_1 \subseteq \widetilde{N}$. Moreover, if $N_1, N_2 \in \mathcal{N}$, there is $N \in \mathcal{N}$ such that $N \subseteq N_1 \cap N_2$ and this implies immediately that $\widetilde{N} \subseteq \widetilde{N}_1 \cap \widetilde{N}_2$. Now, now by Theorm 5, Sec. 2.2 in [1], we have proved that there is a unique vector topology on $\tilde{X}$ with $\widetilde{\mathcal{N}}$ as a zero local base.

If $\tilde{x} = [(x_\lambda)_{\lambda\in I}] \in \widetilde{N}$ for all $N \in \mathcal{N}$, then for a fixed $N \in \mathcal{N}$, there is $\lambda_N \in I$, such that $x_\lambda \in N$, whenever $\lambda_N \leq \lambda$. Thus, $x_\lambda \to 0$ and this implies $\tilde{x}$ is zero. Hence, $\tilde{X}$ is a Hausdorff space. Since $N = \tilde{N} \cap X$ $(\forall N \in \mathcal{N})$, it is follows easily that X is a subspace of $\tilde{X}$. Moreover, if $\tilde{x} = [(x_\alpha)_{\alpha \in I}] \in \tilde{X}$, then we can show that the net $(x_\lambda)_{\lambda \in I} \subseteq X$ converges to $\tilde{x}$ in $\tilde{X}$, thus, X is dense in $\tilde{X}$ and this completes the proof. ■

Theorem 3.2 *The TVS* $\tilde{X}$ *is complete.*

Proof. Let $(\tilde{x}_\nu)_{\nu \in K}$ be a Cauchy net in $\tilde{X}$ and let $N \in \mathcal{N}$. For every $\nu \in K$ we choose and fix a Cauchy net $(u_\lambda)_{\lambda \in I_\nu} \in \tilde{x}_\nu$. If $N \in \mathcal{N}$, then there is $\lambda_0 \in I_\nu$ such that $u_{\lambda_1} - u_{\lambda_2} \in N$, whenever $\lambda_0 \leq \lambda_1, \lambda_0 \leq \lambda_2$. Define $P(N, \nu) = u_{\lambda_0}$.

If $N_1 \supseteq N_2, \nu_1 \leq \nu_2$, then we write $(N_1, \nu_1) \leq (N_2, \nu_2)$. This relation turns $\mathcal{N} \times K$ into a directed set, hence, $(P(N, \nu))_{(N,\nu) \in \mathcal{N} \times K}$ is a net in X.

Now we show that $(P(N, \nu))_{(N,\nu) \in \mathcal{N} \times K}$ is a Cauchy net. If W is a neighborhood of 0, there is $N_0 \in \mathcal{N}$ such that $N_0 + N_0 + N_0 \subseteq W$. There is $\nu_0 \in K$ such that $\tilde{x}_{\nu_1} - \tilde{x}_{\nu_2} \in \tilde{N}_0$, whenever $\nu_0 \leq \nu_1, \nu_0 \leq \nu_2$.

If $(N_0, \nu_0) \leq (N_1, \nu_1)$, then $P(N_1, \nu_1) = u_{\lambda_0}$ for some $\lambda_0 \in I_{\nu_1}$, where $(u_\lambda)_{\lambda \in I_{\nu_1}} \in \tilde{x}_{\nu_1}$ is the net we have fixed in the beginning of this proof. Similarly, if $(N_0, \nu_0) \leq (N_2, \nu_2)$, then $P(N_2, \nu_2) = v_{\mu_0}$ for some $\mu_0 \in I_{\nu_2}$, where $(v_\mu)_{\mu \in I_{\nu_2}}$ is the net we fixed net in $\tilde{x}_{\nu_2}$. Since $\tilde{x}_{\nu_1} - \tilde{x}_{\nu_2} \in \tilde{N}_0$, we can choose $\lambda_1 \in I_{\nu_1}$ and $\mu_1 \in I_{\nu_2}$ such that $\lambda_0 \leq \lambda_1, \mu_0 \leq \mu_1$ and $u_{\lambda_1} - v_{\mu_1} \in N_0$. By the definition of $P(N_1, \nu_1)$ we find that $P(N_1, \nu_1) - u_{\lambda_1} = u_{\lambda_0} - u_{\lambda_1} \in N_1$ and similarly, $v_{\mu_1} - P(N_2, \nu_2) \in N_2$. Therefore

$$P(N_1, \nu_1) - P(N_2, \nu_2) \in N_1 + N_0 + N_2 \subseteq N_0 + N_0 + N_0 \subseteq M,$$

since $N_1 \subseteq N_0$ and $N_2 \subseteq N_0$. Hence, $(P(N, \nu))_{(N,\nu) \in \mathcal{N} \times K}$ is a Cauchy net. Now we denote the equivalence class of this net by $\tilde{y}$ and we will show that $\tilde{y}$ is an accumulation point of $(\tilde{x}_\nu)_{\nu \in K}$ and by Corollary 2.5 we find that $(\tilde{x}_\nu)_{\nu \in K}$ converges to $\tilde{y}$ and this completes the proof.

Let $\nu_0 \in K, M \in \mathcal{N}$ and choose $M_1 \in \mathcal{N}$ such that $M_1 + M_1 \subseteq M$. Since $(P(N, \nu))_{(N,\nu) \in \mathcal{N} \times K}$ is a Cauchy net, there is $(N_1, \nu_1) \in \mathcal{N} \times K$ such that $P(N_1, \nu_1) - P(N, \nu) \in M_1$, whenever $(N_1, \nu_1) \leq (N, \nu)$. We can, and we will, choose $(N_1, \nu_1) \in \mathcal{N} \times K$ such that $N_1 \subseteq M_1$ and $\nu_0 \leq \nu_1$. Let $(u_\lambda)_{\lambda \in I_{\nu_1}}$ be the fixed Cauchy net in the equivalence class $\tilde{x}_{\nu_1}$ defined in the start of this proof. Then $P(N_1, \nu_1) = u_{\lambda_0}$ where $\lambda_0 \in I_{\nu_1}$ and $u_\lambda - u_{\lambda_0} \in N_1$, whenever $\lambda_0 \leq \lambda$. Hence,

$$u_\lambda - P(N, \nu) = u_\lambda - u_{\lambda_0} + P(N_1, \nu_1) - P(N, \nu) \in N_1 + M_1 \subseteq M,$$

whenever $\lambda_0 \leq \lambda$ and $(N_1, \nu_1) \leq (N, \nu)$. Therefore, $\tilde{x}_{\nu_1} - \tilde{y} \in \tilde{M}$. Thus, $\tilde{y}$ is an accumulation of the given Cauchy net. ■

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