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KALMAN FILTER EMPIRICAL OPTIMIZATION

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Article information	Abstract
Key words Kalman Filter; Luenberger State Observer; Optimal State Estima-tion; Empirical Optimi-zation; Stochastic Processes. Received 28 08 2025, Accepted 14 09 2025, Available online 15 09 2025	In this paper, in order to simplify the understanding of the Kalman filter and its applications in practice, the optimization of Kalman filter is done empirically, and compared with the Luenberger observer. Therefore, a series of experiments are done for the range of current and delayed Kalman filter innovation gains with specific process dynamic and stochastic parameters. Also, for the sake of comparison, the same is applied on the observer as well. Where, the stochastic process parameters are the stochastic disturbance or the modeling error covariance (Q), and the measurement error covariance (R). Consequently, the tests of the current Kalman filter showed that the empirically measured optimal Kalman innovation gains are identical to the computations of the Kalman filter algorithms. On the other hand, the tests for the delayed version showed that empirical optimal innovation gains slightly diverge from computations of the algorithms.

I. Introduction

The optimal estimation problem is inherently linked to the method of least squares, which is historically known to go back to the time of Carl Friedrich Gauss publication in 1809, then over the Wiener–Kolmogorov filter, where it is based on frequency response spectrum analysis around 1940 and through the discrete time Kalman filter in 1960 [1], which has led to its implementation in digital computers, and even further to continuous time Kalman-Bucy filter 1961 [2]. For more intensive study of the subject [3], [4], [5] and [6] are suggested. Also the Maybeck series of stochastic models, estimation, and control [7], [8], and [9] are recommended as well as Sorenson [10] for more applications.

Since then until today, there are an immense number of publications about Kalman filter implementation in various industrial applications, for example, [11] uses Kalman filter in estimation of the electrical parameters of power transmission lines, while [12] is in optimization for unscented Kalman filter of an embedded platform. Moreover, the work in [13] presents an efficient tuning framework for Kalman filter parameter optimization.

Now, the target of this paper is to measure the performance of Kalman filter empirically, as done by Alsogkier in [14] and [15], where an intensive empirical optimization of a Luenberger state observer [16] has been done.

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The problem formulation of this paper is to execute a series of experiments with different parameters and conditions to estimate the states of a stochastic process by using the Kalman filter as well as the state observer, and then the Sum of Squared Error (SSE) is computed between the true real state of the process and the estimated state for every single experiment. Note that this type of error is only attainable in digital simulation environments and relatively impossible to acquire in real experiments. In other words, to measure the performance of the estimator the true state value has to be known in order to compute the estimation error. The SSE is used as a performance index to assess the estimations and to measure the optimal estimator performance empirically.
In the following, a brief introduction is given to Kalman filter algorithms in section II, then, section III is about the relationship between the Kalman filter and the state observer. After that, the empirical optimizations of the Kalman filter and the sate observer are presented in section IV. Finally, some conclusions are presented in section V, and the list of references is given at the end.

II. Kalman Filter Algorithm

The Kalman filter is a sort of a discrete observer that can be used to estimate state(s) from measurements of a stochastically disturbed process. It is a general stochastic measurement problem that can be applied in practice to filter noisy sensor measurements of stochastic process states. The real stochastic process, which is to be measured (observed), is defined as

$$x(n+1) = ax(n) + b[u(n) + w(n)], \qquad (1a)$$

$$y(n) = cx(n) + v(n), \qquad (1b)$$

where w and v are normally distributed independent stochastic variables with variances Q and R respectively, and a, b & c are the process parameters, u, x, y are the process known input, the state and the measured output respectively. The time evolutions of the Kalman filter state estimation (prediction) are shown in figure 1.

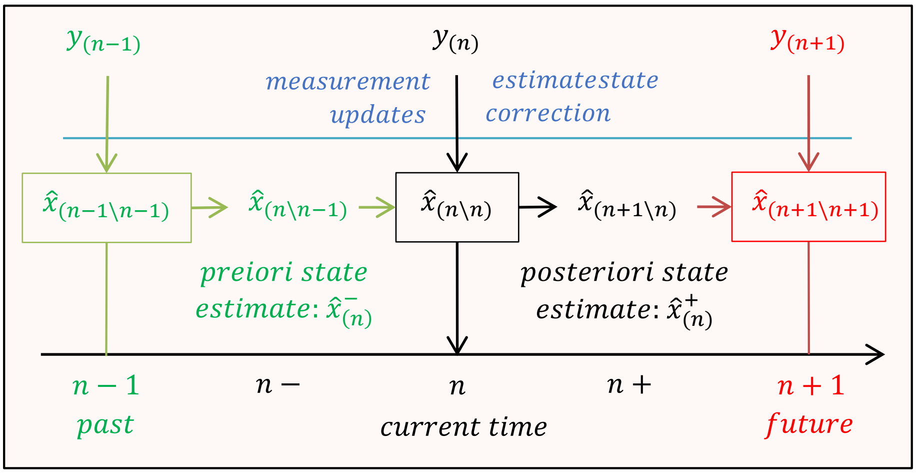

Figure 1: The Kalman filter discrete time evolution

First, the estimation algorithm starts with an initial guess $\hat{x}(0)$ and its variance P(0), the Kalman filter has mainly two phases, time update phase where a prediction is made based on the past data, as well as, the measurement update phase, when a new measurement sample becomes available and used to correct the estimated prediction of time update, that is why, it is called state estimate correction alternatively. The Kalman algorithm can start either by time update first, format A [4], or by measurement update first format B [5], both formats are used and get the same result.

The time update phase is a prediction of prior state estimate based on the previous prediction of the state on the past before the time = n comes. This is described sometimes by n-, and for a simple first order system is defined by

$$\hat{x}(n\backslash n-1) = a\hat{x}(n-1\backslash n-1) + bu(n-1), \tag{2}$$

and its variance propagation equation

$$P(n\backslash n-1) = aP(n-1\backslash n-1)a + bQb. \tag{3}$$

Then at time *n*, the current time, a new measurement sample comes and used to update (correct) the previous (prior) state estimate or prediction, which is usually called measurement update or estimate correction, as following

$$\hat{x}(n\backslash n) = \hat{x}(n\backslash n-1) + K_n[y(n) - c\hat{x}(n\backslash n-1)], \tag{4}$$

where K_n is the innovation gain and its value ranges between 0 to 1, it is a weighting factor that balances the posterior state prediction between the priori estimation and the (new) current measurement update. The variance of the last estimate is given by

$$P(n\backslash n) = K_n R K_n + [1 - K_n c]P(n\backslash n-1)[1 - K_n c]. \tag{5}$$

Now the Kalman gain is chosen to minimize the variance of the last estimate, therefore, the derivative of the last equation is taken and put equal to zero by putting $\frac{dP(n\backslash n)}{dK_n} = 0$, as well as $\frac{dP(n\backslash n-1)}{dK_n} = 0$ yields

$$0 = 2K_n R - 2[1 - K_n c]P(n\backslash n-1)c,$$

and after some mathematical manipulations, the optimal innovation gain, also called the Kalman gain, becomes

$$K_n = \frac{P(n\backslash n-1)c}{cP(n\backslash n-1)c + R} \ , \tag{6}$$

with some precomputations, this result is put back again into the variance equation, which results

$$P(n\backslash n) = \frac{cP(n\backslash n-1)RP(n\backslash n-1)c}{[cP(n\backslash n-1)c + R]^2} + \frac{RP(n\backslash n-1)R}{[cP(n\backslash n-1)c + R]^2}$$

So, after manipulations and substitutions, the variance update becomes

$$P(n\backslash n) = (1 - K_n c)P(n\backslash n-1). \tag{7}$$

The current Kalman filter state is defined as following

$$\hat{x}(n\backslash n) = \hat{x}(n\backslash n-1) + K_n[y(n) - c\hat{x}(n\backslash n-1)] \tag{8a}$$

or

$$\hat{x}(n\backslash n) = [1 - K_n c]\hat{x}(n\backslash n-1) + K_n y(n),$$

and the estimator output without delay is defined as following

$$\hat{y}(n\backslash n) = c\hat{x}(n\backslash n) = c[1 - K_n c]\hat{x}(n\backslash n-1) + cK_n y(n). \tag{8b}$$

While, the delayed output is defined as

$$\hat{y}(n\backslash n-1) = c\hat{x}(n\backslash n-1). \tag{9}$$

The following figure 2 depicts the block diagram of Kalman filter format A algorithm, while table 1 presents the Kalman filter algorithm phases of format A and their recursive equations. Moreover, table 2 presents more detailed steps of Kalman filter format A procedure, where it starts with initial state estimate $\hat{x}(0)$, and its variance $P(0)$, while $u(n)$ is known through all the time [4].

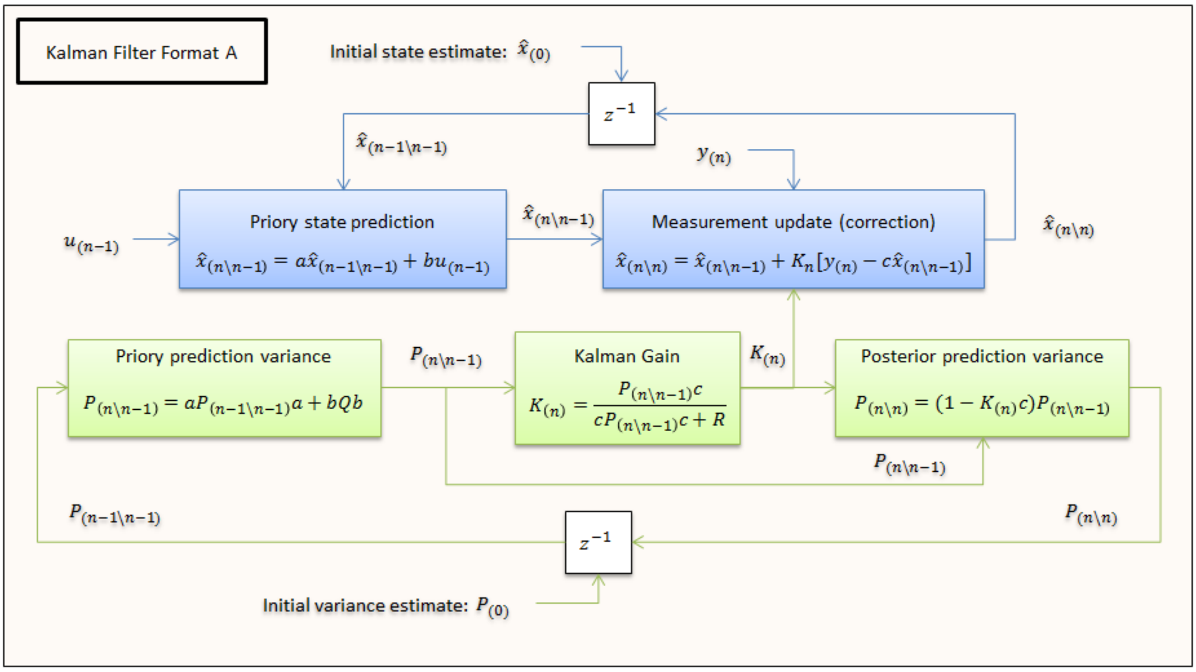

Figure 2: Block diagram of Kalman filter format A algorithm.

Table 1: Kalman filter format A algorithm Phases and their recursive equations.

Time update (at n-) state estimation Before the new sample comes	Measurement update (at n) estimate correction After the new sample [$y(n)$] has come

Initial state estimate $\hat{x}(0)$, and its variance $P(0)$.	$K_n = P(n\backslash n-1)c/(cP(n\backslash n-1)c+R)$
$\hat{x}(n\backslash n-1) = a\hat{x}(n-1\backslash n-1) + bu(n-1)$	$\hat{x}(n\backslash n) = \hat{x}(n\backslash n-1) + K_n[y(n) - c\hat{x}(n\backslash n-1)]$
$P(n\backslash n-1) = aP(n-1\backslash n-1)a + bQb$	$P(n\backslash n) = (1-K_n c)P(n\backslash n-1)$
Wait for the next sample!	Store and Loop!

Table 2: Detailed steps of Kalman filter format A procedure.

n, steps		Command	Comment
0	0	$\hat{x}(0), P(0)$	Initial Conditions
0	1	$\hat{x}(n+1\backslash n) = a\hat{x}(n\backslash n) + bu(n)$	Make prediction for the next sample time
0	2	$P(n+1\backslash n) = aP(n\backslash n)a + bQb$	Update prediction variance
0	3	$K_{n+1} = P(n+1\backslash n)c/[cP(n+1\backslash n)c+R]$	Calculate the next sample correction gain
0	4	$\hat{x}(n\backslash n-1) = \hat{x}(n+1\backslash n); P(n\backslash n-1) = P(n+1\backslash n); K_n = K_{n+1}$	$[z^{-1}]$ Store for the next sample
0	5	Wait for the next sample time to come	
1	0	Read: y(n), and u(n)	New sample hit
1	1	$\hat{x}(n\backslash n) = \hat{x}(n\backslash n-1) + K_n[y(n) - c\hat{x}(n\backslash n-1)]$	Previous estimation correction
1	2	$P(n\backslash n) = (1-K_n c)P(n\backslash n-1)$	Variance of estimation correction
1	3	GOTO: 01	Loop!

On the other hand, Table 3 presents the Kalman filter algorithm phases of format B and their recursive equations, while figure 3 depicts the block diagram of Kalman filter format B algorithm. Moreover, table 4 presents more detailed steps of Kalman filter format B procedure, where it starts with initial state estimate $\hat{x}(0)$, and its variance $P(0)$, while $u(n)$ is known through all the time [5].

Table 3: Kalman filter format B algorithm Phases and their recursive equations.

Starting with initial state estimate $\hat{x}(0)$, and its variance $P(0)$, u(0), y(0).	
Measurement update (at n) estimate correction after the new sample has come	Time update (at n+) state estimation prediction for the next sample
$K_n = P(n\backslash n-1)c/(cP(n\backslash n-1)c+R)$	$\hat{x}(n+1\backslash n) = a\hat{x}(n\backslash n) + bu(n)$
$\hat{x}(n\backslash n) = \hat{x}(n\backslash n-1) + K_n[y(n) - c\hat{x}(n\backslash n-1)]$	$P(n+1\backslash n) = aP(n\backslash n)a + bQb$
$P(n\backslash n) = (1-K_n c)P(n\backslash n-1)$	Store and go to the next measurement

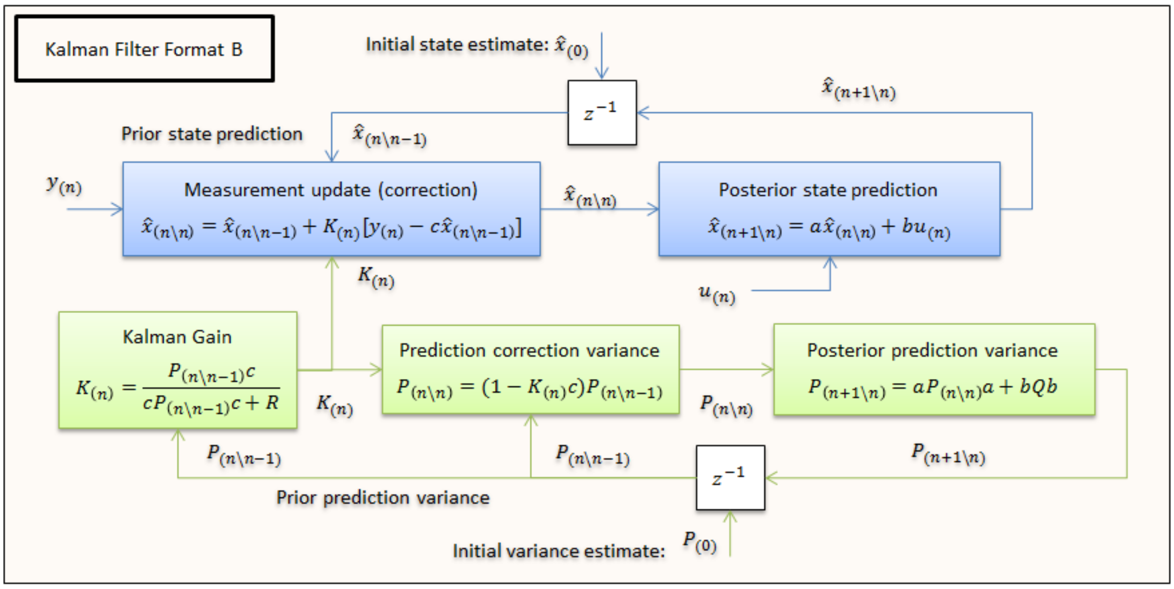

Figure 3: Block diagram of Kalman filter format B algorithm.

Table 4: Detailed steps of Kalman filter format B procedure.

n, steps		Command	Comment
0	0	$\hat{x}(0), P(0)$, y(0), u(0)	Initial Conditions
0	1	$K_n = P(n\backslash n-1)c/(cP(n\backslash n-1)c + R)$	Calculate the correction gain
0	2	$\hat{x}(n\backslash n) = \hat{x}(n\backslash n-1) + K_n[y(n) - c\hat{x}(n\backslash n-1)]$	Previous estimation correction with new sample
0	3	$P(n\backslash n) = (1 - K_n c)P(n\backslash n-1)$	Correction variance
0	4	$\hat{x}(n+1\backslash n) = a\hat{x}(n\backslash n) + bu(n)$	Make prediction for the next sample time
0	5	$P(n+1\backslash n) = aP(n\backslash n)a + bQb$	next sample prediction variance
0	6	$\hat{x}(n\backslash n-1) \Leftarrow \hat{x}(n+1\backslash n); P(n\backslash n-1) \Leftarrow P(n+1\backslash n)$	$[z^{-1}]$ Store for the next sample
0	7	Wait for the next sample time to come	
1	0	Read: y(n), and u(n)	New sample hit
1	1	GOTO: 0,1	Loop!

III. Relationship between Kalman and the State Observer

A. Reformation of Kalman filter format A to an observer:

Combine the time update equation (2) with the measurement update equation (4) as well as current output with past real state equation (1) as following

$$\hat{x}(n\backslash n) = a\hat{x}(n-1\backslash n-1) + bu(n-1)$$
$$+K_n c[ax(n-1) + bu(n-1)$$
$$-a\hat{x}(n-1\backslash n-1) - bu(n-1)].$$

Dropping the conditional probability slash

$$\hat{x}(n) = a\hat{x}(n-1) + bu(n-1)$$
$$+K_n c[ax(n-1) + bu(n-1) - a\hat{x}(n-1) - bu(n-1)],$$

which yields

$$\hat{x}(n) = a\hat{x}(n-1) + bu(n-1)$$
$$+K_n ca[x(n-1) - \hat{x}(n-1)],$$

advance one step ahead for both sides, yields Kalman filter reformed like an observer:

$$\hat{x}(n+1) = a\hat{x}(n) + bu(n) + K_n ca[x(n) - \hat{x}(n)] \quad (10)$$

By comparing to the observer standard equation

$$\hat{x}_o(n+1) = a\hat{x}_o(n) + bu(n) + K_o c[x(n) - \hat{x}_o(n)], \quad (11)$$

where $\hat{x}_o$ is estimated state and the observer gain is defined by

$$K_o = K_n a \,,$$

or by using the outputs instead of the states in equation (11)

$$\hat{x}_o(n+1) = a\hat{x}_o(n) + bu(n) + K_o[y(n) - \hat{y}_o(n)]$$
$$\hat{y}_o(n) = c\hat{x}_o(n) \quad (12)$$

also

$$\hat{x}_o(n+1) = [a - K_o c]\, \hat{x}_o(n) + b\, u(n) + K_o\, y(n).$$

B. Reformation of Kalman filter format B to an observer:

Starting by substitution of the measurement update equation (4) into the time update equation (2), yields

$$\hat{x}(n+1\backslash n) = a\hat{x}(n\backslash n-1) + aK_n[y(n) - c\hat{x}(n\backslash n-1)] + bu(n)$$

or

$$\hat{x}(n+1\backslash n) = a\hat{x}(n\backslash n-1) + bu(n) + aK_n[y(n) - \hat{y}(n\backslash n-1)].$$

By dropping the conditional probability slash, then the Kalman filter reformed like an observer becomes

$$\hat{x}(n+1) = a\hat{x}(n) + bu(n) + aK_n[y(n) - \hat{y}(n)],$$

which is like the observer equation (10) of format A.

IV. Empirical Optimization Tests for the Kalman Filter and the Observer

A. Experimental Setup

In this section, in order to measure the optimal performance of the Kalman filter empirically, a series of experimental tests are done for a range of Kalman innovation gain values with different system parameters and conditions. Where the performance measure is defined by the sum of squared (true) error between the true state and the estimated one, and computed for every single test run that corresponds to the Kalman gain, as following

$$J(K_{MAN}) = \sum_{n=0}^{N} e_{true}^2(n),$$

where, K_{MAN} is a value of Kalman innovation gain (K_n) which is set constant for every single experimental run, N is the number of samples and e_{true} is the error between the true and the estimated states.

Then, for a series of tests, the performance measure is plotted against their corresponding gains (K_{MAN}) and then the optimal gain is visually found out from the plotted curve, which corresponds to minimal value of the SSE.

The first series of tests are conducted on the current Kalman filter to measure the Kalman optimal innovation gain empirically. Figure 4, presents the performance measure curves for a set of different parameters and conditions as presented in table 5. While in the second series of tests, the same set of the parameters and conditions as in the first case are repeated to optimize the current (un delayed) observer gain empirically. Also, Figure 5 presents the performance measure curves for the current observer.

Again, the first series is redone again for the delayed Kalman filter, its curves are presented in figure 6. While the second series is also redone again for the case of the delayed observer as well, its empirical optimization curves are presented in figure 7.

At the end all of the empirically measured optimal gains, for all cases and series are summarized and presented in table 5.

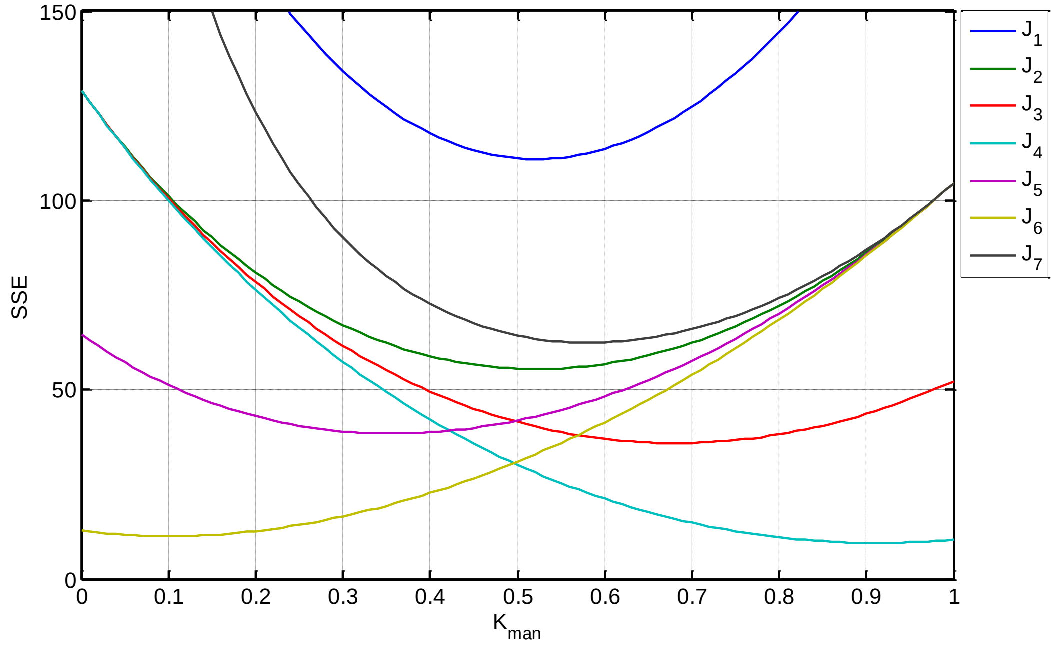

Figure 4. Empirical optimization for current Kalman filter.

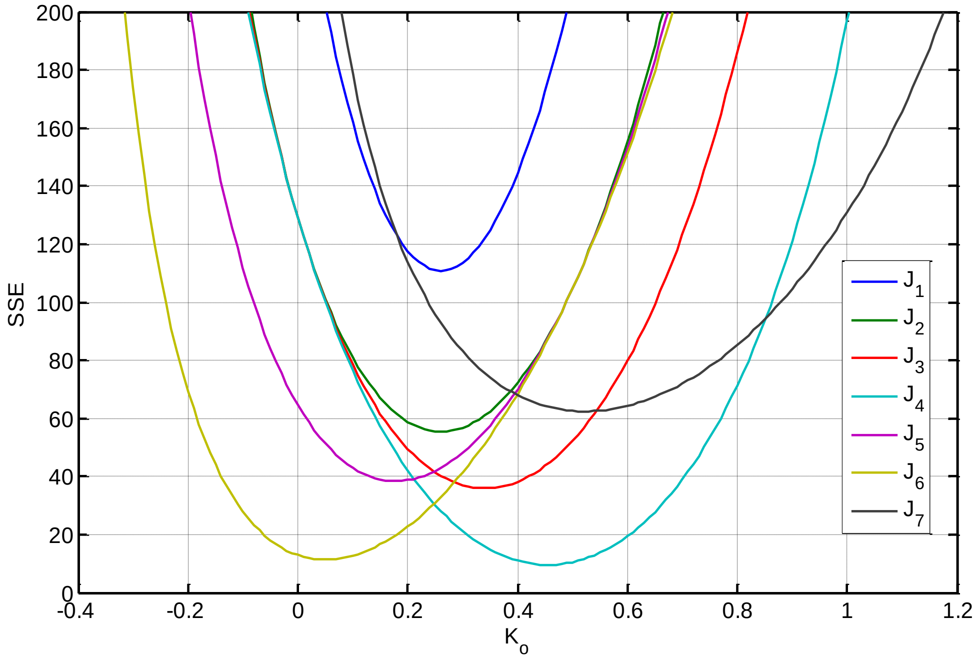

Figure 5. Empirical optimization for current observer.

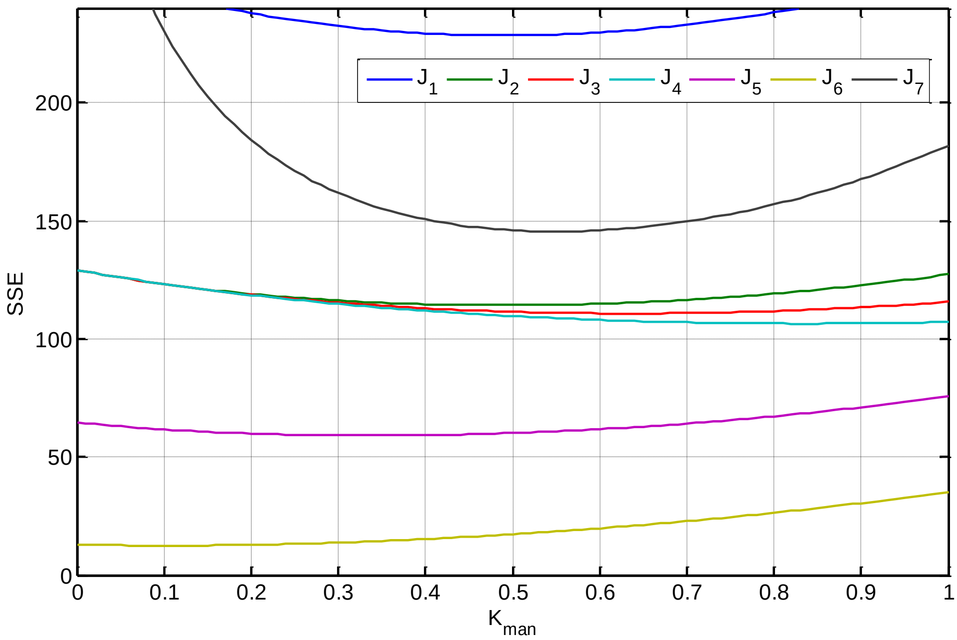

Figure 6. Empirical optimization for delayed Kalman filter.

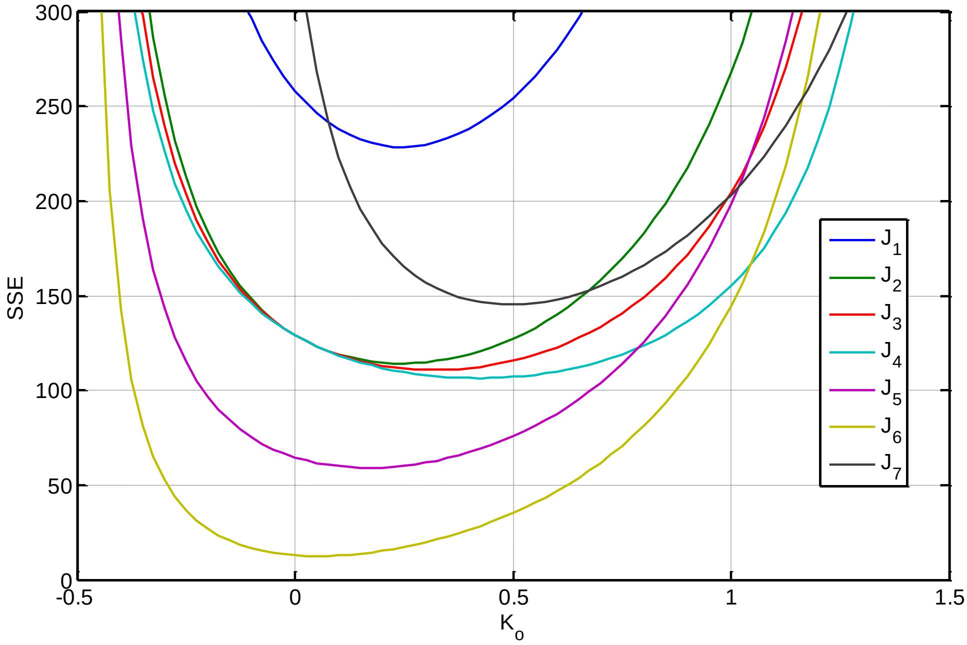

Figure 7. Empirical optimization for delayed observer.

Table 5: System parameters, computed gains and empirically measured optimal gains.

SSE	system parameters						computed		current empirical		delayed empirical	
	a	b	c	d	Q	R	K_{MAN}	K_O	K_{MAN}	K_O	K_{MAN}	K_O
J_1	0.5	1	1	0	0.2	0.2	0.53311	0.2656	0.52	0.26	0.49	0.25
J_2	0.5	1	1	0	0.1	0.1	0.53311	0.2656	0.52	0.26	0.49	0.25
J_3	0.5	1	1	0	0.1	0.05	0.6847	0.3423	0.68	0.34	0.635	0.325
J_4	0.5	1	1	0	0.1	0.01	0.9109	0.4555	0.91	0.455	0.84	0.425
J_5	0.5	1	1	0	0.05	0.1	0.3723	0.1861	0.36	0.18	0.34	0.175
J_6	0.5	1	1	0	0.01	0.1	0.1138	0.0569	0.1	0.05	0.105	0.05
J_7	0.9	1	1	0	0.1	0.1	0.5974	0.5377	0.58	0.52	0.55	0.5

B. Results Discussion

From the table 5 and figure 4, J_1 and J_2 have the same parameters and Q/R ratio, but only the variance is different. Nevertheless, the optimal gains, computed and empirical, are identical since the Q/R ratio is equal. By comparing J_2 and J_3, in J_3 the measurement variance is less than in case of J_2, therefore, the information in the measurement is more important than in the model, therefore it is weighted more by higher innovation and observer gains. In the J_4 case, the measurement variance becomes much less than in the J_3 case, therefore, the gains are greater than in J_3 and J_2. J_5 and J_6 cases are exactly opposite to J_3 and J_4, where the variance of the model is less than the measurement variance, therefore, the correction gains in J_5 are smaller than in J_2 and J_3. Furthermore, in J_6 the gains are even smaller than in the case of J_5, since the variance ratio Q/R is less than in all other cases. In the last case J_7 shows the impact of the system dynamics of the gains in comparison to J_2 case.

In addition, for the case of J_2 parameters, the time responses of the measured, true and estimated outputs are plotted for a different Kalman innovation gain values, in figure 8 is equal to 0.1, figure 9 equal to the optimal and figure 10 equal to 0.9. Moreover, the true error between the true and estimated outputs and the measured error between the measured and the estimated outputs are also plotted. In figure 8 the innovation gain is very small that the estimation depends more on the model, this is why the estimation looks like less noisy, while in figure 10 the gain is almost one, here the estimation depends heavily on the measurements and this is the cause why the measured error is very small, but in figure 9 the sum of squared true error is at minimum, see also curve J_2 in figure 4.

It is clear, that the empirical measurement of Kalman filter optimal gains, the innovation as well as the observer gains have matched the computed optimal gains particularly in the current Kalman filter algorithm or in the un delayed observer form. Nevertheless, on the other hand, the delayed version of Kalman filter form, the empirical optimal measurements of the Kalman innovation gain as well as the observer gain show slight deviations in comparison to the current computed one, see figures 6 and 7. The current (un delayed) Kalman filter and observer functions are identical and they perform better than the delayed versions, but in practice, these are unrealizable (very difficult, almost impossible) particularly in real time control applications.

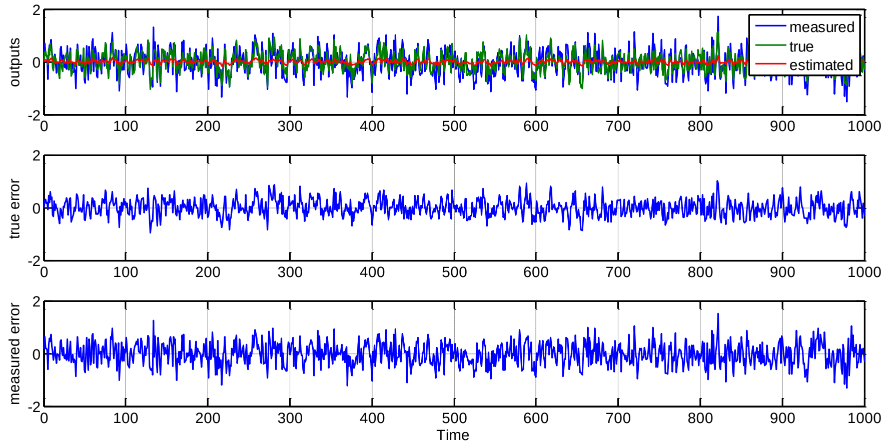

Figure 8. Outputs and errors for Kalman gain equal to 0.1.

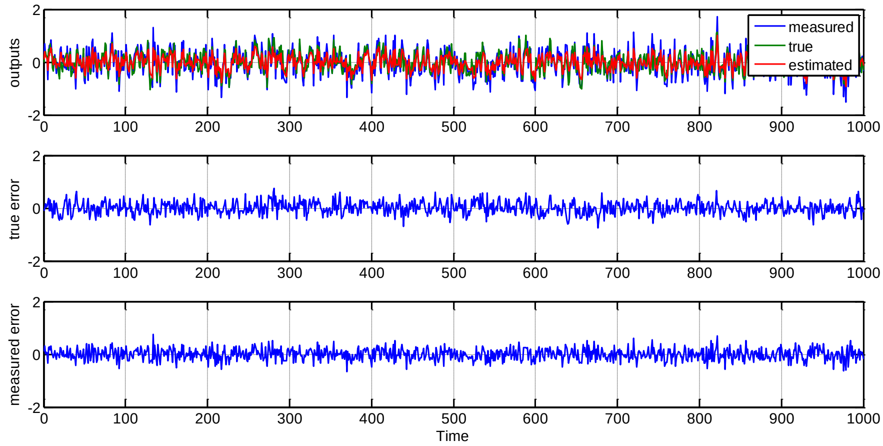

Figure 9. Outputs and errors for the optimal Kalman gain (0.53311).

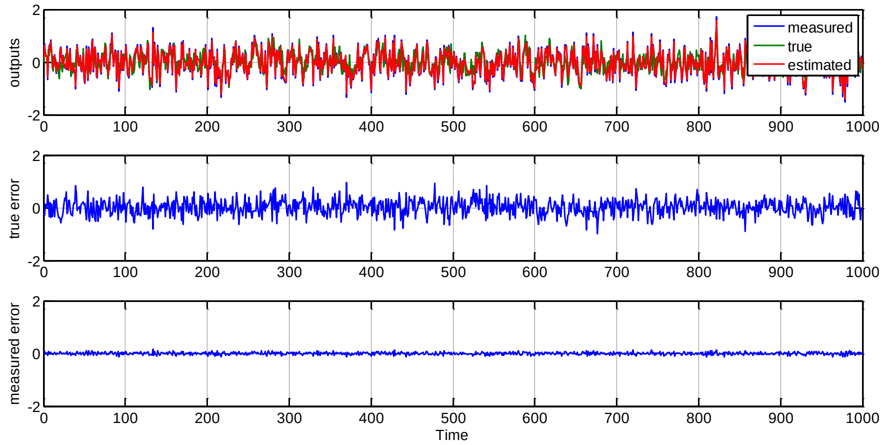

Figure 10. Outputs and errors for Kalman gain equal to 0.9.

V. CONCLUSIONS

Both Kalman filter formats A and B are identical. Also, the current Kalman filter algorithms can be combined to work exactly as a state observer.

The empirical optimization tests of the current Kalman filter, in all of its formats, showed that the empirically measured optimal Kalman innovation gains are identical to the computations of the Kalman filter algorithms.

On the other hand, the empirical optimization tests for the delayed Kalman filter version, for all formats, showed that the empirically measured optimal Kalman innovation gains slightly diverge from the optimal computations of Kalman filter algorithms.

Therefore, it can be stated, that the delayed Kalman filter version is a suboptimal version of the current Kalman filter algorithms. This means, for the delayed version, it has to be redesigned to be optimal. The delayed version can be easily implemented in real time applications on the contrary to the current version, which is impossible to realize in real time applications.

The optimal design of Kalman filter does not depend on the absolute values of modeling uncertainty or disturbance variance and the measurement error variance but rather on the ratio between them. This is a practical advantage since the tuning of the filter can be reduced down to one parameter instead of two, particularly, for example, when modelling uncertainty or the disturbance is unknown, while the measurement variance is known.

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